



## UNIT-III PARTIAL DIFFERENTIAL EQUATIONS Solution of Second Order Partial Differential Equations

Linear PDE of 2nd and higher order with constant coefficients.

Homogeneous Linear PDEs:

A linear PDE with constant coefficients in which all the partial derivatives are of the same order is called homogeneous; otherwise it is called non-homogeneous.

Example:

Homogeneous Equation:

$$\frac{\partial^2 z}{\partial x^2} + 5 \frac{\partial^2 z}{\partial x \partial y} + 6 \frac{\partial^2 z}{\partial y^2} = \sin x$$

Non-homogeneous Equation:

$$\frac{\partial^2 z}{\partial x^2} - 5 \frac{\partial z}{\partial x} + 7 \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} = e^{x+y}$$

Notation:

$$D = \frac{\partial}{\partial x}, \quad D' = \frac{\partial}{\partial y}$$

Method of finding complementary function (CF)

Let the given equation be of the form

$$f(D, D')z = f(x, y).$$



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Put  $D = m$   
 $D' = 1$

$$f(m, 1) = 0 \Rightarrow a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$$

Let the roots of the eqn. be  $m_1, m_2, \dots, m_n$

Roots	Complementary function
1. The roots are different. $m_1, m_2, \dots, m_n$	$CF = f_1(y + m_1 x) + f_2(y + m_2 x) + \dots + f_n(y + m_n x)$
2. The roots are equal. $m_1 = m_2 = \dots = m_p = m$ (say)	$CF = f_1(y + m x) + x f_2(y + m x) + \dots + x^{p-1} f_p(y + m x)$

General solution is  $y = CF + PI$

RHS = 0 ( $x = CF$ )

1. Solve  $(D^2 - 6DD' + 9D'^2)x = 0$

Soln.:

Put  $D = m, D' = 1$

The auxiliary equation is,

$$m^2 - 6m + 9 = 0$$

$$(m - 3)(m - 3) = 0$$

$$m = 3, 3 \text{ (equal roots)}$$

The solution is

$$x = CF = f_1(y + 3x) + x f_2(y + 3x)$$

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$$\text{RHS} = e^{ax+by}$$

Replace  $D$  by  $a$   
 $D'$  by  $b$

Q. Solve  $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

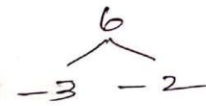
Soln:

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \quad (D \rightarrow m, D' \rightarrow 1)$$

$$(m-3)(m-2) = 0$$

$$m = 2, 3 \text{ (Different)}$$



$$\text{CF} = f_1(y+2x) + f_2(y+3x)$$

$$\text{PI} = \frac{1}{D^2 - 5DD' + 6D'^2} e^{x+y}$$

$$= \frac{1}{1-5+6} e^{x+y}$$

Replace  
 $D \rightarrow a = 1$   
 $D' \rightarrow b = 1$

$$= \frac{1}{2} e^{x+y}$$

The solution is,  $z = \text{CF} + \text{PI}$   
 $z = f_1(y+2x) + f_2(y+3x) + \frac{e^{x+y}}{2}$

Q. Solve  $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

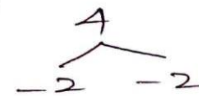
Soln:

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$m = 2, 2 \text{ (Equal)}$$



$$\text{CF} = f_1(y+2x) + x f_2(y+2x)$$



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$$\begin{aligned}
 \text{PI} &= \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y} \\
 &= \frac{1}{2^2 - 4(2)(1) + 4(1)^2} e^{2x+y} \quad \begin{array}{l} \text{Replace} \\ D \rightarrow a=2 \\ D' \rightarrow b=1 \end{array} \\
 &= \frac{1}{4 - 8 + 4} e^{2x+y} \quad [\text{multiply } x \text{ in the No. \& differentiate w.r.t } D] \\
 &= x \frac{1}{2D - 4D'} e^{2x+y} \\
 &= x^2 \frac{1}{2} e^{2x+y} \quad \begin{array}{l} D \rightarrow 2 \\ D' \rightarrow 1 \end{array} \\
 &= \frac{x^2}{2} e^{2x+y}
 \end{aligned}$$

The solution is  $x = \text{CF} + \text{PI}$

$$\begin{aligned}
 &= y_1(y+2x) + x y_2(y+2x) \\
 &\quad + \frac{x^2}{2} e^{2x+y}
 \end{aligned}$$

HW Solve

$$\text{i]. } \frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} = e^{2x-y}$$

2]. Solve Find the PI of

$$(D^2 + DD')z = e^{x-y} + e^{x+y}$$



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$$\text{RHS} = \cos(ax+by) \text{ or } \sin(ax+by)$$

Replace  $D^2 \rightarrow -a^2$

$$DD' \rightarrow -ab$$

$$D'^2 \rightarrow -b^2$$

1. Solve  $(D^2 - 2DD' + D'^2)z = \cos(x-3y)$

Soln.:

AE  $m^2 - 2m + 1 = 0$

$$(m-1)(m-1) = 0$$

$$m = 1, 1 \text{ (equal)}$$

$$\text{CF} = f_1(y+x) + x f_2(y+x)$$

PI =  $\frac{1}{D^2 - 2DD' + D'^2} \cos(x-3y)$   $a=1, b=-3$

$$= \frac{1}{-1 - 2(-3) - 9} \cos(x-3y)$$

$$= \frac{1}{16} \cos(x-3y)$$

$$\begin{aligned} D^2 &\rightarrow -a^2 = -1^2 = -1 \\ DD' &\rightarrow -ab = -1(-3) = 3 \\ D'^2 &\rightarrow -b^2 = -(-3)^2 = -9 \end{aligned}$$

$$\therefore z = \text{CF} + \text{PI}$$

$$= f_1(y+x) + x f_2(y+x) - \frac{1}{16} \cos(x-3y)$$

2. Solve  $(D^2 - 4D'^2)z = \sin(2x+y)$

Soln.:

AE  $m^2 - 4 = 0$

$$(m+2)(m-2) = 0$$

$$m = -2, 2 \text{ (different)}$$

$$\text{CF} = f_1(y-2x) + f_2(y+2x)$$



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$$PI = \frac{1}{D^2 - 4D + 2} \sin(2x+y)$$

$$a=2$$

$$b=1$$

$$D^2 \rightarrow -a^2 = -2^2$$

$$= -4$$

$$= \frac{1}{-4 - 4(-1)} \sin(2x+y)$$

$$DD' \rightarrow -ab = -2(1)$$

$$= -2$$

$$= x \frac{1}{2D} \sin(2x+y)$$

$$D'^2 \rightarrow -b^2 = -1^2$$

$$= -1$$

$$= \frac{x}{2} \left( \frac{-\cos(2x+y)}{2} \right)$$

$$PI = -\frac{x \cos(2x+y)}{4}$$

∴ The soln. is  $Z = CF + PI$

$$Z = f_1(y-2x) + f_2(y+2x) - \frac{x \cos(2x+y)}{4}$$

3. Find the PI of  $(D^2 - 3DD' + D'^2)z = \sin x \cos y$

Soln.:

$$PI = \frac{1}{D^2 - 3DD' + D'^2} \sin x \cos y$$

$$\text{Gm. } (D^2 - 3DD' + D'^2)z = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$$

$$PI = \frac{1}{2} \left[ \frac{1}{D^2 - 3DD' + D'^2} \sin(x+y) \right]$$

$$+ \frac{1}{D^2 - 3DD' + D'^2} \sin(x-y)$$

$$= \frac{1}{2} [PI_1 + PI_2] \rightarrow (1)$$

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$$\begin{aligned}
 PI_1 &= \frac{1}{D^2 - 3DD' + D'^2} \sin(x+y) & a=1, b=1 \\
 &= \frac{1}{1 - 3(-1) - 1} \sin(x+y) & D^2 \rightarrow -a^2 = -1 \\
 &= \frac{1}{-2+3} \sin(x+y) & DD' \rightarrow -ab = -1 \\
 & & D'^2 \rightarrow -b^2 = -1 \\
 &= \sin(x+y)
 \end{aligned}$$

$$\begin{aligned}
 PI_2 &= \frac{1}{D^2 - 3DD' + D'^2} \sin(x-y) & a=1, b=-1 \\
 &= \frac{1}{1 - 3(1) - 1} \sin(x-y) & D^2 \rightarrow -a^2 = -1 \\
 & & DD' \rightarrow -ab = -1(-1) = 1 \\
 & & D'^2 \rightarrow -b^2 = -(-1)^2 = -1 \\
 &= \frac{1}{-5} \sin(x-y)
 \end{aligned}$$

$$\begin{aligned}
 (1) \Rightarrow PI &= \frac{1}{2} \left[ \sin(x+y) - \frac{1}{5} \sin(x-y) \right] \\
 &= \frac{1}{2} \sin(x+y) - \frac{1}{10} \sin(x-y)
 \end{aligned}$$

Q. Find the PI of  $(D^2 + 4DD' - 5D'^2)z = \sin(x-2y)$

Sol:

$$\begin{aligned}
 PI &= \frac{1}{D^2 + 4DD' - 5D'^2} \sin(x-2y) & a=1, b=-2 \\
 &= \frac{1}{1 + 4(2) - 5(-4)} \sin(x-2y) & D^2 \rightarrow -a^2 = -1 \\
 &= \frac{1}{29} \sin(x-2y) & DD' \rightarrow -ab = -1(-2) = 2 \\
 & & D'^2 \rightarrow -b^2 = -(-2)^2 = -4
 \end{aligned}$$

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$$\text{RHS} = x^m y^n$$

J. Solve  $(D^2 - 4DD' + 4D'^2) z = xy$

Soln.

$$\underline{\text{AE}} \quad m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$m = 2, 2 \text{ (Equal)}$$

$$\text{CF} = \delta_1 (y + 2x) + x\delta_2 (y + 2x)$$

$$\text{PI} = \frac{1}{D^2 - 4DD' + 4D'^2} xy$$

$$= \frac{1}{D^2 \left[ 1 - \frac{4DD'}{D^2} + \frac{4D'^2}{D^2} \right]} xy$$

$$= \frac{1}{D^2 \left[ 1 - \left( \frac{4D'}{D} - \frac{4D'^2}{D^2} \right) \right]} xy$$

$$= \frac{1}{D^2 \left[ 1 - \left( \frac{4D'}{D} - \frac{4D'^2}{D^2} \right) \right]^{-1}} xy$$

$$= \frac{1}{D^2 \left[ 1 + \left( \frac{4D'}{D} - \frac{4D'^2}{D^2} \right) + \dots \right]} xy$$

$$(\because (1-x)^{-1} = 1 + x + x^2 + \dots)$$





$$= \frac{1}{D^2} \left[ xy + \frac{4D'}{D}(xy) - 0 \right]$$

$$= \frac{1}{D^2} \left[ xy + \frac{4}{D} x \right]$$

$$= \frac{1}{D^2} xy + \frac{4}{D^3} x$$

$$= \frac{x^3 y}{6} + 4 \frac{x^4}{24}$$

$$= \frac{x^3 y}{6} + \frac{x^4}{6}$$

$$\frac{1}{D^2} xy \xrightarrow{1^{st}} \frac{1}{D} \frac{x^2}{2} y \rightarrow \frac{x^3}{6} y$$

$$\frac{1}{D^3} x \rightarrow \frac{1}{D^2} \frac{x^2}{2} \rightarrow \frac{1}{D} \frac{x^3}{6} \rightarrow \frac{x^4}{24}$$

∴ The solution is,  $x = CF + PI$

$$= f_1(y+2x) + x f_2(y+2x) + \frac{x^3 y}{6} + \frac{x^4}{6}$$

2) Find the PI of  $(D^2 - DD' - 2D'^2)x = 2x + 3y$

Soln. :

$$PI = \frac{1}{D^2 - DD' - 2D'^2} (2x + 3y)$$

$$= \frac{1}{D^2} \left[ 1 - \frac{D'}{D} - \frac{2D'^2}{D^2} \right]^{-1} (2x + 3y)$$

$$= \frac{1}{D^2} \left[ 1 - \left( \frac{D'}{D} + \frac{2D'^2}{D^2} \right) \right]^{-1} (2x + 3y)$$

$$= \frac{1}{D^2} \left[ 1 + \left( \frac{D'}{D} + \frac{2D'^2}{D^2} \right) + \dots \right] (2x + 3y)$$

$$= \frac{1}{D^2} \left[ 2x + 3y + \frac{D'}{D} (2x + 3y) \right]$$



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$$= \frac{1}{D^2} [12x + 3y] + \frac{1}{D} [3]$$

$$= \frac{1}{D^2} [(2x + 3y) + \frac{3}{D}]$$

$$= \frac{1}{D^2} (2x + 3y) + \frac{3}{D^3}$$

$$\begin{aligned} \frac{1}{D^2} (2x + 3y) &= \frac{1}{D} \left[ \frac{2x^2}{2} + 3xy \right] \\ &= \frac{x^3}{3} + \frac{3x^2y}{2} \end{aligned}$$

$$\frac{1}{D^3} = \frac{1}{D^2} x = \frac{1}{D} \frac{x^2}{2} = \frac{x^3}{6}$$

$$= \frac{x^3}{3} + \frac{3x^2y}{2} + 3 \frac{x^3}{6}$$

$$\text{PI} = \frac{x^3}{3} + \frac{3x^2y}{2} + \frac{x^3}{2}$$

$$\text{RHS} = e^{ax+by} + \sin(ax+by)$$

$$e^{ax+by} + \cos(ax+by)$$

ii. Solve  $(D^2 - DD' - 20D'^2)x = e^{5x+y} + \sin(4x-y)$

Soln

$$\text{AE } m^2 - m - 20 = 0$$

$$(m+5)(m-4) = 0$$

$$m = 5, -4$$

$$\text{CF} = \delta_1 (y - 4x) + \delta_2 (y + 5x)$$

$$\text{PI} = \frac{1}{D^2 - DD' - 20D'^2} [e^{5x+y} + \sin(4x-y)]$$

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$$= \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y} + \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$$

$$PI = PI_1 + PI_2$$

$$PI_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}$$

$$= \frac{1}{25 - 5(1) - 20(1)^2} e^{5x+y} \quad \begin{array}{l} D \rightarrow a = 5 \\ D' \rightarrow b = 1 \end{array}$$

$$= \frac{1}{0} e^{5x+y}$$

$$= x \frac{1}{2D - D'} e^{5x+y}$$

$$= x \frac{1}{2(5) - 1} e^{5x+y}$$

$$PI_1 = \frac{x}{9} e^{5x+y}$$

$$PI_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y) \quad \begin{array}{l} a = 4, b = -1 \\ D^2 \rightarrow a^2 = -16 \end{array}$$

$$= \frac{1}{-16 - 4 - 20(-1)} \sin(4x-y) \quad \begin{array}{l} DD' \rightarrow -ab = -4(-1) \\ = 4 \end{array}$$

$$= x \frac{1}{2D - D'} \sin(4x-y) \quad \begin{array}{l} D'^2 \rightarrow -b^2 = -(-1)^2 \\ = -1 \end{array}$$

$$= x \frac{(2D + D')}{(2D - D')(2D + D')} \sin(4x-y)$$

$$= x \frac{(2D + D') \sin(4x-y)}{4D^2 - D'^2}$$

$$= \frac{x}{-64 + 1} (2D + D') \sin(4x-y)$$

$$= -\frac{x}{63} [2D \sin(4x-y) + D' \sin(4x-y)]$$

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$$= -\frac{x}{63} [8 \cos(4x-y) - \cos(4x-y)]$$

$$= -\frac{7x}{63} \cos(4x-y)$$

$$= -\frac{x}{9} \cos(4x-y)$$

The soln. is,  $z = CF + PE$

$$z = f_1(y-4x) + f_2(y+5x) + \frac{x}{9} e^{5x+y} - \frac{x}{9} \cos(4x-y)$$

4.  $(D^2 + 4DD' - 5D'^2)z = e^{2x-y} + \sin(x-2y)$

2].  $(D^2 - DD' - 20D'^2)z = xy + e^{6x+y}$

Solve  $x + 3 - 6t = y \cos x$

Soln:

Given  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + 6 \frac{\partial^2 z}{\partial y^2} = y \cos x$

$$(D^2 + DD' - 6D'^2)z = y \cos x$$

AE

$$m^2 + m - 6 = 0$$

$$(m+3)(m-2) = 0$$

$$m = -3, 2$$

$$CF = f_1(y-3x) + f_2(y+2x)$$

$$PI = \frac{1}{(D^2 + DD' - 6D'^2)} y \cos x$$



$$\begin{aligned} & \text{factor} \rightarrow D - 2D' \\ & \text{where } y = C - 2x \\ & D \rightarrow C \\ & D' \rightarrow x \\ & = \frac{1}{(D+3D')(D-2D')} y \cos x \\ & = \frac{1}{(D+3D')} \int (C-2x) \cos x \, dx \\ & = \frac{1}{D+3D'} [ (C-2x) \sin x - (-2)(-\cos x) ] \\ & = \frac{1}{D+3D'} [ y \sin x - 2 \cos x ] \quad \text{factor} \rightarrow D+3D' \\ & \quad \quad \quad y \rightarrow C+3x \\ & = \int [ (C+3x) \sin x - 2 \cos x ] \, dx \\ & = (C+3x)(-\cos x) - 3(-\sin x) - 2 \sin x \\ & = -y \cos x + 3 \sin x - 2 \sin x \\ & = -y \cos x + \sin x \end{aligned}$$

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