Solution of Standard types of first order PDEs.

A Partial differential Equation in which the partial derivatives occur are of the first degree, is said to be linear; otherwise it is said to be non-linear.

Standard Types:

Type 1: $F(p, q) = 0$

Type 2: $z = px + qy + f(p, q)$ [Clairut's form]

Type 3: $f(z, p, q) = 0$

Type 4: $f_1(x, p) = f_2(y, q)$

Type I Working Rule

1). Let $z = ax + by + c$ be the complete integral

$$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b$$

2). Put $b = \phi(a)$ for general solution.

3). There is no singular integral.

Solve $p + q = pq \rightarrow (1)$

Soln.:

Let $z = ax + by + c \rightarrow (2)$

Complete Integral:

Diff. partially w.r.to 'x' and 'y'

$$\frac{\partial z}{\partial x} = a \quad \Big| \quad \frac{\partial z}{\partial y} = b$$

$$p = a \quad \Big| \quad q = b$$

Subs. the above values in (1), we get

$$a+b = ab$$
$$a = ab - b$$
$$a = b(a-1) \Rightarrow b = \frac{a}{a-1}$$

The complete integral is,

$$z = ax + \left(\frac{a}{a-1}\right)y + c \rightarrow (3)$$

Singular Integral:

Diff. (3) partially w.r.t a and c and equal to zero.

$$\frac{\partial z}{\partial a} = x + \left[\frac{(a-1)(1) - a(1)}{(a-1)^2}\right] y = 0$$

$$\frac{\partial z}{\partial c} = 1 \neq 0$$

There is no singular Integral.

General Integral:

put $c = \phi(a)$ in (3)

$$z = ax + \left(\frac{a}{a-1}\right)y + \phi(a) \rightarrow (4)$$

Diff. (4) partially w.r.t 'a'

$$\frac{\partial z}{\partial a} = x + \left[\frac{(a-1)(1) - a(1)}{(a-1)^2}\right] y + \phi'(a) = 0 \rightarrow (5)$$

Eliminate 'a' b/w (4) and (5), we get the general solution.

2). Solve $\sqrt{p} + \sqrt{q} = 1$

Soln.: $\sqrt{p} + \sqrt{q} = 1 \rightarrow (1)$

Let $z = ax + by + c$

Complete Integral:

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b$$

Subs. the above values in (1), we get

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = (1-\sqrt{a})^2$$

The complete integral is,

$$z = ax + (1-\sqrt{a})^2 y + c \rightarrow (2)$$

Singular Integral:

$$\frac{\partial z}{\partial a} = \cancel{x} + 2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right) y = 0$$

$$\frac{\partial z}{\partial c} = 1 \neq 0$$

There is no singular integral.

General Integral:

Put $c = \phi(a)$ in (2)

$$z = ax + (1-\sqrt{a})^2 y + \phi(a) \rightarrow (3)$$

Diff. (3) partially w.r.t 'a'

$$\frac{\partial z}{\partial a} = x + 2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right) y + \phi'(a) = 0 \rightarrow (4)$$

Eliminate 'a' b/w (3) and (4),

we get the general Integral.

HW 1). Find the C.I. of $p - q = 0$

2). Solve $p^2 + q^2 - 4pq = 0$