Lagrange's Linear Equation:

The equation is of the form

$Pp + Qq = R$, where P, Q and R are functions of x, y, z. This is known as Lagrange's linear eqn.

To solve this equation, It is enough to solve the subsidary (or) auxillary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The auxillary eqn. can be solved in two ways.

i). method of grouping
ii) method of multipliers.

Method of grouping:

In the auxillary eqn., $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

If the variables can be separated in any pair of eqns., then we get a solution is of the form

$u(x, y) = c_1$ and $v(x, y) = c_2$

i.e., $\phi(u, v) = 0$ where ϕ is arbitrary.

I. Solve $px^2 + qy^2 = z^2$

Soln.:

$px^2 + qy^2 = z^2$

This eqn. is of the form

$Pp + Qq = R$ where $P = x^2$, $Q = y^2$ and $R = z^2$

The auxillary eqn. is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

Take $\frac{dx}{x^2} = \frac{dy}{y^2}$

Integrating, we get

$$\int x^{-2}\,dx = \int y^{-2}\,dy$$

$$\frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + c_1$$

$$-\frac{1}{x} + \frac{1}{y} = c_1$$

$$u = \frac{1}{y} - \frac{1}{x}$$

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

$$\int y^{-2}\,dy = \int z^{-2}\,dz$$

$$\frac{y^{-1}}{-1} = \frac{z^{-1}}{-1} + c_2$$

$$\frac{-1}{y} + \frac{1}{z} = c_2$$

$$v = \frac{1}{z} - \frac{1}{y}$$

$\therefore$ The soln. is $\phi(u, v) = 0$

$$\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0$$

3. Solve $\frac{y^2 z}{x} p + xzq = y^2$

Soln.:

$$\frac{y^2 z}{x} p + xzq = y^2 \qquad P = \frac{y^2 z}{x}, \quad Q = xz$$

$$R = y^2$$

AE

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz}$$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$$

$$x^2\,dx = y^2\,dy$$

Integrating,

$$\frac{x^3}{3} = \frac{y^3}{3} + c_3$$

$$\frac{x^3}{3} - \frac{y^3}{3} = c_3$$

$$x^3 - y^3 = 3c_3 = u$$

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dz}{y^2}$$

$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$$

$$x\,dx = z\,dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c_2$$

$$x^2 - z^2 = 2c_2$$

$$v = x^2 - z^2$$

$\therefore$ The soln. is $\phi(u, v) = 0$

$$\phi(x^3 - y^3,\ x^2 - z^2) = 0$$

3. Solve $px + qy = x$

Soln.

$Pp + Qq = R \Rightarrow P = x, \quad Q = y, \quad R = x$

AE $\quad \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{x}$

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating,

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1 \Rightarrow u = \frac{x}{y}$$

$$\frac{dx}{x} = \frac{dz}{x}$$

$$dx = dz$$

$$\int dx = \int dz$$

$$x = z + c_2$$

$$x - z = c_2$$

$$\Rightarrow v = x - z$$

$$\Rightarrow \phi\left(\frac{x}{y},\ x - z\right) = 0$$

4]. Solve $p\tan x + q\tan y = \tan z$

Soln.:

$$Pp + Qq = R \Rightarrow P = \tan x, \quad Q = \tan y, \quad R = \tan z$$

AE $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Integrating,

$$\int \cot x\, dx = \int \cot y\, dy$$

$$\log(\sin x) = \log(\sin y) + \log c_1$$

$$\log(\sin x) - \log(\sin y) = \log c_1$$

$$\log\left(\frac{\sin x}{\sin y}\right) = \log c_1$$

$$\frac{\sin x}{\sin y} = c_1$$

$$\Rightarrow u = \frac{\sin x}{\sin y}$$

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\int \cot y\, dy = \int \cot z\, dz$$

$$\log(\sin y) = \log(\sin z) + \log c_2$$

$$\log(\sin y) - \log(\sin z) = \log c_2$$

$$\log\left(\frac{\sin y}{\sin z}\right) = \log c_2$$

$$\frac{\sin y}{\sin z} = c_2$$

$$\Rightarrow v = \frac{\sin y}{\sin z}$$

$$\Rightarrow \phi(u, v) = 0$$

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

5]. Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Soln.:

$$Pp + Qq = R \Rightarrow P = \sqrt{x}, \quad Q = \sqrt{y}, \quad R = \sqrt{z}$$

AE $\dfrac{dx}{\sqrt{x}} = \dfrac{dy}{\sqrt{y}} = \dfrac{dz}{\sqrt{z}}$

$$\int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$$

$$2\sqrt{x} = 2\sqrt{y} + 2c_1$$

$$\sqrt{x} = \sqrt{y} + c_1$$

$$\sqrt{x} - \sqrt{y} = c_1$$

$$\Rightarrow \quad u = \sqrt{x} - \sqrt{y}$$

$$\int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$$

$$2\sqrt{y} = 2\sqrt{z} + 2c_2$$

$$\sqrt{y} = \sqrt{z} + c_2$$

$$\sqrt{y} - \sqrt{z} = c_2$$

$$\Rightarrow \quad v = \sqrt{y} - \sqrt{z}$$

$$\Rightarrow \quad \phi(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$$

HW

1]. $px + qy = \sqrt{z}$

2]. $x^2 p + y^2 q = z^2$

3]. $p - q = \log(x + y)$