UNIT-III   PARTIAL DIFFERENTIAL EQUATIONS   Solution of First Order Partial Differential Equations

TYPE-II   claireaut's Form $z = px + qy + f(p, q)$

Working Rule:

Complete Integral:

Replace $p \to a$ and $q \to b$

Singular Integral:

$$\frac{\partial z}{\partial a} = 0 \quad \text{and} \quad \frac{\partial z}{\partial b} = 0$$

General Integral:

Put $b = \phi(a)$ in Complete Integral.

I. Solve $z = px + qy + pq$

Soln. :

Gryn. $z = px + qy + pq \to (1)$

Complete Integral :

$$z = ax + by + ab \quad \left[\text{Replace } p \to a \atop q \to b \right]$$
$$\hookrightarrow (2)$$

Singular Integral :

$$\frac{\partial z}{\partial a} = 0 \quad \text{and} \quad \frac{\partial z}{\partial b} = 0$$

$$x + b = 0 \qquad\qquad y + a = 0$$
$$b = -x \qquad\qquad a = -y$$

Subs. a & b in (2)

$$z = -yx - xy - y(-x)$$
$$z = -xy$$

General Integral :

Subs. $b = \phi(a)$ in (2),
$$z = ax + \phi(a) y + a\,\phi(a) \longrightarrow (3)$$

WKT $\frac{\partial z}{\partial a} = 0$

$$\Rightarrow x + \phi'(a) y + a \phi'(a) + \phi(a) = 0 \to (4)$$

Eliminate 'a' b/w (4) & (3), we get the general Soln.

Solve $z = px + qy + p^2 - q^2$

Soln.:

Givn. $z = px + qy + p^2 - q^2 \longrightarrow$ (1)

Complete Integral :

$z = ax + by + a^2 - b^2$ Replace $p \rightarrow a$
$\qquad\qquad\qquad \searrow$ (2) $\qquad q \rightarrow b$

Singular Integral :

$\dfrac{\partial z}{\partial a} = 0$ $\qquad\qquad\qquad$ $\dfrac{\partial z}{\partial b} = 0$

$x + 2a = 0$ $\qquad\qquad\qquad$ $y - 2b = 0$

$2a = -x$ $\qquad\qquad\qquad\qquad$ $y = 2b$

$a = \dfrac{-x}{2}$ $\qquad\qquad\qquad \Rightarrow\; b = \dfrac{y}{2}$

Subs. a & b in (2),

$z = -\dfrac{x}{2}\,x + \dfrac{y}{2}\,y + \left(\dfrac{-x}{2}\right)^2 - \left(\dfrac{y}{2}\right)^2$

$\quad = -\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{x^2}{4} - \dfrac{y^2}{4}$

$4z = -2x^2 + 2y^2 + x^2 - y^2$

$\quad = -x^2 + y^2$

$y^2 - x^2 = 4z$

General Integral :

Subs. $b = \phi(a)$ in (2)

$z = ax + \phi(a)y + [\phi(a)]^2 + a^2 \longrightarrow$ (3)

WKT $\dfrac{\partial z}{\partial a} = 0$

$\Rightarrow x + \phi'(a)y - 2\phi(a)\phi'(a) + 2a = 0 \longrightarrow$ (4)

Eliminate 'a' b/w (3) & (4), we
get the general soln.

2]. Solve $z = px + qy + \sqrt{1+p^2+q^2}$

Sol$\underline{n}$. :

Giv$\underline{n}$. $z = px + qy + \sqrt{1+p^2+q^2}$

Complete Integral :

$$z = ax + by + \sqrt{1+a^2+b^2} \longrightarrow (A)$$

Singular Integral :

$$\frac{\partial z}{\partial a} = 0 \qquad\qquad\qquad \frac{\partial z}{\partial b} = 0$$

$$x + \frac{1 \, (2a)}{2\sqrt{1+a^2+b^2}} = 0 \qquad\qquad y + \frac{1 \, (2b)}{2\sqrt{1+a^2+b^2}} = 0$$

$$x = \frac{-a}{\sqrt{1+a^2+b^2}} \longrightarrow (1) \qquad\qquad y = \frac{-b}{\sqrt{1+a^2+b^2}} \longrightarrow (2)$$

Squaring on both sides,

$$x^2 = \frac{a^2}{1+a^2+b^2} \qquad\qquad y^2 = \frac{b^2}{1+a^2+b^2}$$

Now,

$$x^2 + y^2 = \frac{a^2 + b^2}{1+a^2+b^2}$$

$$1 - (x^2 + y^2) = 1 - \frac{a^2 + b^2}{1+a^2+b^2}$$

$$1 - x^2 - y^2 = \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2} = \frac{1}{1+a^2+b^2}$$

Taking square root,

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{1+a^2+b^2}} \implies \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

UNIT-III   PARTIAL DIFFERENTIAL EQUATIONS   Solution of First Order Partial Differential Equations

$(1) \Rightarrow \quad x = -a\sqrt{1-x^2-y^2} \qquad \Rightarrow \quad a = \dfrac{-x}{\sqrt{1-x^2-y^2}}$

$(2) \Rightarrow \quad y = -b\sqrt{1-x^2-y^2} \qquad \Rightarrow \quad b = \dfrac{-y}{\sqrt{1-x^2-y^2}}$

$(A) \Rightarrow \quad z = \dfrac{-x^2}{\sqrt{1-x^2-y^2}} - \dfrac{y^2}{\sqrt{1-x^2-y^2}} + \dfrac{1}{\sqrt{1-x^2-y^2}}$

$$= \dfrac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$= \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

HW   1. $\quad z = px + qy + (pq)^{3/2}$

      2. $\quad z = px + qy + p^2 q^2$