Method of multipliers:

choose any 3 multiplier l, m, n which may be constant (or) functions of x, y, z, then

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$, then $l\,dx + m\,dy + n\,dz = 0$

Direct integration gives $u(x, y, z) = C_1$

Illy choose another set of 3 multipliers l', m' and n',

$v(x, y, z) = C_2$

$\Rightarrow \phi(u, v) = 0$

J. Solve $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$

Soln. :

$x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2) \rightarrow (1)$

This eqn. is of the form,

$Pp + Qq = R$

where $P = x(z^2-y^2)$; $Q = y(x^2-z^2)$; $R = z(y^2-x^2)$

AE $\dfrac{dx}{x(z^2-y^2)} = \dfrac{dy}{y(x^2-z^2)} = \dfrac{dz}{z(y^2-x^2)}$

choosing x, y, z as lagrange's multipliers,

$$\frac{x\,dx + y\,dy + z\,dz}{x^2(z^2-y^2) + y^2(x^2-z^2) + z^2(y^2-x^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2z^2 - x^2y^2 + y^2x^2 - y^2z^2 + y^2z^2 - z^2x^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

ie, $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$x^2 + y^2 + z^2 = 2c_1$$

$\Rightarrow u = x^2 + y^2 + z^2$

choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as lagrange's multipliers

$$\frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{\frac{1}{x}\,x(z^2-y^2) + \frac{1}{y}\,y(x^2-z^2) + \frac{1}{z}\,z(y^2-x^2)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{z^2 - y^2 + x^2 - z^2 + y^2 - x^2}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

ie, $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

Integrating, we get

$\log x + \log y + \log z = \log c_2$

$\log(xyz) = \log c_2$

$\Rightarrow c_2 = xyz$

$\Rightarrow v = xyz$

The solution is, $\phi(u, v) = 0$

$\phi(x^2+y^2+z^2, xyz) = 0$

2]. Solve $(mz-ny)p + (nx-lz)q = ly-mx$

Soln.:

$(mz-ny)p + (nx-lz)q = ly-mx \rightarrow (1)$

The eqn. is of the form

$Pp + Qq = R$

where $P = (mz-ny)$; $Q = (nx-lz)$; $R = ly-mx$

AE $\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$

choosing l, m, n as lagrange's multipliers,

$$\frac{l\,dx + m\,dy + n\,dz}{l(mz-ny) + m(nx-lz) + n(ly-mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{lmz - lny + mnx - lmz + lny - mnx}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

ie, $l\,dx + m\,dy + n\,dz = 0$

Integrating, we get

$$lx + my + nz = c_1$$

$$\Rightarrow u = lx + my + nz$$

choosing x, y, z as lagrange's multipliers, we get

$$\frac{x\,dx + y\,dy + z\,dz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{mxz - nxy + nxy - lyz + lyz - mxz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

ie, $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$x^2 + y^2 + z^2 = 2c_1$$

$$\Rightarrow v = x^2 + y^2 + z^2$$

The solution is, $\phi(u, v) = 0$

$$\phi(lx + my + nz,\ x^2 + y^2 + z^2) = 0$$

3]. Solve $(3z-4y)p+(4x-2z)q = 2y-3x$

Soln. :

This eqn. is of the form

$$Pp+Qq=R$$

Here $P = 3z-4y$; $Q = 4x-2z$; $R = 2y-3x$

AE $$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}$$

choosing 2, 3, 4 as lagrange's multipliers,

$$\frac{2dx+3dy+4dz}{2(3z-4y)+3(4x-2z)+4(2y-3x)}$$

$$= \frac{2dx+3dy+4dz}{6z-8y+12x-6z+8y-12x}$$

$$= \frac{2dx+3dy+4dz}{0}$$

ie, $2dx+3dy+4dz=0$

Integrating, we get

$$2x+3y+4z = c_1$$

$$\Rightarrow u = 2x+3y+4z$$

choosing x, y, z as lagrange's multipliers,

$$\frac{xdx+ydy+zdz}{x(3z-4y)+y(4x-2z)+z(2y-3x)}$$

$$= \frac{xdx+ydy+zdz}{3xz-4xy+4xy-2yz+2yz-3xz}$$

$$= \frac{xdx+ydy+zdz}{0}$$

ie, $xdx+ydy+zdz=0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$x^2 + y^2 + z^2 = 2c_2$$

$$\Rightarrow v = x^2 + y^2 + z^2$$

The solution is, $\phi(u, v) = 0$

$$\phi(2x + 3y + 4z,\ x^2 + y^2 + z^2) = 0$$

4). Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Soln. :

$$Pp + Qq = R$$

Here $P = x(y^2+z)$; $Q = -y(x^2+z)$;
$R = z(x^2-y^2)$

AE $\dfrac{dx}{x(y^2+z)} = \dfrac{dy}{-y(x^2+z)} = \dfrac{dz}{z(x^2-y^2)} \rightarrow (2)$

choosing x, y and -1 as lagrange's multipliers, each ratio of (2) is equal to

$$\frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)}$$

$$= \frac{x\,dx + y\,dy - dz}{x^2y^2 + x^2z - x^2y^2 - y^2z - x^2z + y^2z}$$

$$= \frac{x\,dx + y\,dy - dz}{0}$$

ie, $x\,dx + y\,dy - dz = 0$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - z = c_1$$

$$x^2 + y^2 - 2z = 2c_1$$

$$\text{i.e., } u = x^2 + y^2 - 2z$$

choosing $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$ as lagrange's multipliers,

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{\frac{1}{x}x(y^2+z) - \frac{1}{y}y(x^2+z) + \frac{1}{z}z(x^2-y^2)}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y^2+z-x^2-z+x^2-y^2}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\text{i.e., } \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2$$

$$\Rightarrow c_2 = xyz$$

$$\Rightarrow v = xyz$$

The solution is,

$$\phi(u, v) = 0$$

$$\phi(x^2+y^2-2z, xyz) = 0$$