UNIT-1

INTRODUCTION

Finite Element Analysis (FEA)

FEA is a numerical method that predicts the behaviour of products subjected to loads.

Variation Methods

1. Ritz method
2. Rayleigh Ritz method

Static analysis

The solution of the problem does not vary with respect to time.

Dynamic analysis

The solution of the problem varies with respect to time

Eg. Vibration analysis problem

Weighted - Residual methods

1. Point Collection method.
2. Sub domain method.
3. Least square method.
4. Galerkins method.

General Steps to solve the Finite Element Problems

1. Discretization of the element
2. Numbering of notes and elements
3. Selection of a displacement function
4. Define the material behaviour using strain displacement and stress strain relationship
5. Derivation of Element stiffness matrix and elevation (Governing equations)

6. Assemble the elements stiffness matrix
7. Apply the boundary conditions
8. Solution for the unknown displacement
9. Computations of the element strains and stresses from the nodal displacement
10. Data interpretation

1. The following differential equation is available for a physical phenomenon $\frac{d^2y}{dx^2} + 50 = 0, \quad 0 \le x \le 10$

Trial function

$$y = a_1 x (10 - x)$$

Boundary functions are $y(0) = 0$, $y(10) = 0$

Find the value of the parameter a_1 by the following methods.

1. Point co-location method
2. Sub domain co-location method
3. Least square method
4. Galerkins method

Point collocation method

$$R = 0$$

The residuals set to be zero

Sub domain collocation method

$$\int_0^1 R_i \, dx$$

The weighting function choosen

Least square method

$$I = \int R^2 dx$$

(or)

$$\frac{\partial I}{\partial a_1} = \int R \frac{\partial R}{\partial a_1} dx$$

Galerkins method

$$\int W_i R \, dx = 0$$

$$W_i = N_i(x) = \text{Trial function}$$

Solution

$y(0) = a_1(0)(10-0)$

$y(0) = 0$

$y(10) = a_1(10)(10-10)$

$y(10) = 0$

$\therefore$ The given boundary condition satisfies with trial function

1. Point Collocation method

$y = a_1 x(10-x)$

$y = 10a_1 x - a_1 x^2$

$\frac{dy}{dx} = 10a_1 - 2a_1 x$

$\frac{d^2y}{dx^2} = -2a$

So, the residuals

$\frac{d^2y}{dx^2} + 50 = 0$

$-2a_1 + 50 = 0$

$a_1 = 25$

Trial function,

$y = 25x(10-x)$

2. Sub domain Collocation method

$\int_0^{10} R.dx = 0$

$\int_0^{10} (-2a_1 + 50)\, dx = 0$

$-2a_1 (x)_0^{10} + 50(x)_0^{10} = 0$

$-2a_1(10-0) + 50(10-0) = 0$

$-20a_1 + 500 = 0$

$-20a_1 = -500$

$a_1 = 25$

3. Least square method

$$I = \int R^2 \, dx \quad \text{(or)} \quad \frac{\partial I}{\partial a_1} = \int R \frac{\partial R}{\partial a_1} \, dx = 0$$

$$R = -2a_1 + 50$$

$$\frac{\partial R}{\partial a_1} = -2$$

$$\frac{\partial I}{\partial a_1} = \int_0^{10} (-2a_1 + 50) \times (-2) \, dx$$

$$0 = \int_0^{10} (4a_1 - 100) \, dx$$

$$0 = 4a_1 (x)_0^{10} - 100 (x)_0^{10}$$

$$0 = 4a_1 (10) - 100 (10)$$

$$0 = 40a_1 - 1000$$

$$a_1 = 25$$

4. Galerkins method

$$\int_0^{10} W_i \, R \, dx = 0$$

$$\int_0^{10} a_1 x (10 - x) \times (-2a_1 + 50) \, dx = 0$$

$$\int_0^{10} (10a_1 x - a_1 x^2)(-2a_1 + 50) \, dx = 0$$

$$(-2a_1 + 50) \int_0^{10} (10a_1 x - a_1 x^2) \, dx = 0$$

$$(-2a_1 + 50) \; 10a_1 (x)_0^{10} - 2a_1 (x)_0^{10} = 0$$

$$(-2a_1 + 50)(100a_1 - 20a_1) = 0$$

$$(-2a_1 + 50)(80a_1) = 0$$

$$-160a_1^2 + 4000a_1 = 0$$

$$-160a_1^2 = -4000a_1$$

$$160a_1 = 4000$$

$$a_1 = \frac{4000}{160}$$

$$a_1 = 25$$

2. The following differential eqn is $\frac{d^2y}{dx^2} + 500 = 0, \ 0 \le x \le 10$.

Trial function, $y = a_1 x^2 (10-x)$

Boundary condition, $y(0) = 0$, $y(10) = 0$

Solution

$y = a_1 x^2 (10-x)$

$y(0) = a_1(0)(10-0)$

$y(0) = 0$

$y(10) = a_1(10)(10-10)$

$y(10) = 0$

The given boundary condition is satisfied.

1. Point Collocation method

$y = a_1 x^2 (10-x)$

$y = 10a_1 x^2 - a_1 x^3$

$\frac{dy}{dx} = 20a_1 x - 3a_1 x^2$

$\frac{d^2y}{dx^2} = 20a_1 - 6a_1 x$

Residuals, $\frac{d^2y}{dx^2} + 500 = 0$

$20a_1 - 6a_1 x + 500 = 0$

$20a_1 - 6a_1(0.5) + 500 = 0$ [Assume $x = 0.5$]

$20a_1 - 3a_1 = -500$

$a_1 = -29.41$

2. Subdomain collocation method

$\int_0^{10} R \, dx = 0$

$\int_0^{10} (17a_1 + 500) \, dx = 0$

$17a_1 (x)_0^{10} + 500 (x)_0^{10} = 0$

$170 a_1 + 5000 = 0$

$a_1 = -29.41$

3. Least square method

$$I = \int R^2 dx \quad (\text{or}) \quad \frac{\partial I}{\partial a_1} = \int R \frac{\partial R}{\partial a_1} dx = 0$$

$$R = 17a_1 + 500$$

$$\frac{\partial R}{\partial a_1} = 17$$

$$\frac{\partial I}{\partial a_1} = \int_0^{10} (17a_1 + 500) \times (17)\, dx$$

$$0 = \int_0^{10} (289\, a_1 + 8500)\, dx$$

$$0 = 289\, a_1 (x)_0^{10} + 8500\, (x)_0^{10}$$

$$0 = 289\, a_1 (10-0) + 8500\, (10-0)$$

$$0 = 2890\, a_1 + 85000$$

$$a_1 = -29.41$$

4. Galerkins method

$$\int_0^{10} W_i \, R \, dx = 0$$

$$\int_0^{10} \left[a_1 x^2 (10-x)\right] \left[17a_1 + 500\right] dx = 0$$

$$17a_1 + 500 \int_0^{10} a_1 x^2 (10-x)\, dx = 0$$

$$(17\, a_1 + 500) \int_0^{10} (10\, a_1 x^2 - a_1 x^3)\, dx$$

$$(17a_1 + 500)\left[20\, a_1 (x)_0^{10} - 3\, a_1 (x^2)_0^{10}\right] = 0$$

$$(17\, a_1 + 500)\left[20 a_1 (10) - 3\, a_1 (10^2)\right] = 0$$

$$(17a_1 + 500)(200\, a_1 - 300\, a_1) = 0$$

$$(17\, a_1 + 500)(-100\, a_1) = 0$$

$$-1700\, a_1^2 - 50000\, a_1 = 0$$

$$-1700\, a_1^2 = 50000\, a_1$$

$$-a_1 = \frac{50000}{1700}$$

$$a_1 = -29.41$$

3. The differential equation, $\frac{d^2y}{dx^2} + 500x^2 = 0, \ 0 \le x \le 1.$

Trial function, $y = a_1(x-x^3) + a_2(x-x^5)$

Boundary condition, $y(0)=0, \ y(1)=0$.

Calculate the values of parameters a_1 & a_2 using,

i) Point Collocation ii) Subdomain Point collocation

iii) Least square iv) Galerkins method

Solution

$y = a_1(x-x^3) + a_2(x-x^5)$

$y(0) = a_1(0-0) + a_2(0-0)$

$y(0) = 0$

$y(1) = a_1(1-1) + a_2(1-1)$

$y(1) = 0$

$\therefore$ The given boundary condition is satisfied

(i) Point Collocation method

$y = a_1(x-x^3) + a_2(x-x^5)$

$= a_1x - ax^3 + a_2x - a_2x^5$

$\frac{dy}{dx} = a_1 - 3a_1x^2 + a_2 - 5a_2x^4$

$\frac{d^2y}{dx^2} = -6a_1x - 20a_2x^3$

Residual,

$R = \frac{d^2y}{dx^2} + 500x^2$

$= -6a_1x - 20a_2x^3 + 500x^2$

The interval 0 to 1 is divided into two domains 0 to 1/2 and 1/2 to 1

Domain (i)

Limit 0 to 1/2 $(x = 1/3)$

$0 = R = -6a_1\left(\frac{1}{3}\right) - 20a_2\left(\frac{1}{3}\right)^3 + 500\left(\frac{1}{3}\right)^2$

$0 = -2a_1 - 0.740a_2 + 55.55$

Domain (ii)

Limit $1/2$ to 1 $\left(x = \frac{2}{3}\right)$

$$R = 0 = -6\,a_1\left(\frac{2}{3}\right) - 20\,a_2\left(\frac{2}{3}\right)^3 + 500\left(\frac{2}{3}\right)^2$$

$$0 = -4a_1 - 5.925\,a_2 + 222.22$$

$$a_1 = 18.52$$

$$a_2 = 24.99$$

Trial function,

$$y = 18.52\,(x - x^3) + 24.99\,(x - x^5)$$

(ii) Subdomain Collocation method

$$\int R \cdot dx = 0$$

$$\int_0^1 \left(-6a_1x - 20\,a_2\,x^3 + 500\,x^2\right) dx = 0$$

Domain (i)

$$\int_0^{1/2} \left(-6\,a_1x - 20a_2\,x^3 + 500\,x^2\right) dx = 0$$

$$-6a_1\left(\frac{x^2}{2}\right)_0^{1/2} - 20\,a_2\left(\frac{x^4}{4}\right)_0^{1/2} + 500\left(\frac{x^3}{3}\right)_0^{1/2} = 0$$

$$-6\,a_1\left(\frac{(1/2)^2}{2} - \frac{0^2}{2}\right) - 20\,a_2\left(\frac{(1/2)^4}{4} - \frac{0^4}{4}\right) + 500\left(\frac{(1/2)^3}{3} - \frac{0^3}{3}\right) = 0$$

$$-0.75\,a_1 - 0.312\,a_2 + 20.83 = 0$$

Domain (ii)

$$\int_{1/2}^{1} \left(-6a_1x - 20\,a_2\,x^3 + 500\,x^2\right) dx = 0$$

$$-6a_1\left(\frac{x^2}{2}\right)_{1/2}^{1} - 20\,a_2\left(\frac{x^4}{4}\right)_{1/2}^{1} + 500\left(\frac{x^3}{3}\right)_{1/2}^{1} = 0$$

$$-6\,a_1\left[\frac{(1/2)^2}{2} + \frac{(1/2)^2}{2}\right] - 20\,a_2\left[\frac{(1)^4}{4} - \frac{(1/2)^4}{4}\right] + 500\left[\frac{(1)^3}{3} - \frac{(1/2)^3}{3}\right] = 0$$

$$-2.25\,a_1 - 4.687 + 145.83 = 0$$

$$a_1 = 17.53$$

$$a_2 = 24.61$$

Trial function,

$$y = 17.53\,(x - x^3) + 24.61\,(x - x^5)$$

(iii) Least square method

$$I = \int_0^1 R^2 \, dx = 0$$

$$\frac{\partial I}{\partial a} = \int_0^1 R \frac{\partial R}{\partial a} \, dx = 0$$

Domain (i)

$$R = -6a_1 x - 20 a_2 x^3 + 500 x^2$$

$$\frac{\partial R}{\partial a_1} = -6x$$

$$\frac{\partial I}{\partial a_1} = \int_0^{1/2} (-6a_1 x - 20 a_2 x^3 + 500 x^2) \times (-6x) \, dx = 0$$

$$= \left[+36 a_1 x^2 + 120 a_2 x^4 + (-3000 x^3) \right]_0^{1/2}$$

$$= 36 a_1 \left[\frac{(1/2)^3}{3} \right] + 120 a_2 \left[\frac{(1/2)^5}{5} \right] - 3000 \left[\frac{(1/2)^4}{4} \right]$$

$$0 = 1.5 a_1 + 0.75 a_2 - 46.875$$

Domain (ii)

$$\frac{\partial I}{\partial a_2} = \int_{1/2}^{1} (36 a_1 x^2 + 120 a_2 x^4 - 3000 x^3) \, dx$$

$$= 36 a_1 \left[\frac{(1)^3}{3} - \frac{(1/2)^3}{3} \right] + 120 a_2 \left[\frac{(1)^5}{5} - \frac{(1/2)^5}{5} \right] - 3000 \left[\frac{(1)^4}{4} - \frac{(1/2)^4}{4} \right]$$

$$\frac{\partial I}{\partial a_2} = \int_{1/2}^{1} R \frac{\partial R}{\partial a_2} \, dx = 0$$

$$R = -6a_1 x - 20 a_2 x^3 + 500 x^2$$

$$\frac{\partial R}{\partial a_2} = -20 x^3$$

$$\frac{\partial I}{\partial a_2} = \int_{1/2}^{1} (-6a_1 x - 20 a_2 x^3 + 500 x^2)(-20 x^3) \, dx = 0$$

$$= \int_{1/2}^{1} (120 a_1 x^4 + 400 a_2 x^6 + 10000 x^5) = 0$$

$$0 = 120 a_1 \left[\frac{(1)^5}{5} - \frac{(1/2)^5}{5} \right] + 400 a_2 \left[\frac{(1)^7}{7} - \frac{(1/2)^7}{7} \right] - 10000 \left[\frac{(1)^6}{6} - \frac{(1/2)^6}{6} \right]$$

$$0 = 23.25 a_1 + 56.69 a_2 - 1640.6 = 0$$

$$a_1 = 21$$

$$a_2 = 20$$

(iv) Galerkins method

$$\int_0^1 W_i \times R \, dx = 0$$

Domain (i)

$$\int_0^{1/2} \left[a_1(x-x^3) + a_2(x-x^5)\right]\left[-6a_1x - 20a_2x^3 + 500x^2\right] dx = 0$$

$$\left[a_1\left(\frac{x^2}{2}-\frac{x^4}{4}\right) + a_2\left(\frac{x^2}{2}-\frac{x^6}{6}\right)\right]_0^{1/2}\left[-6a_1\left(\frac{x^2}{2}\right) - 20a_2\left(\frac{x^4}{4}\right) + 500\left(\frac{x^3}{3}\right)\right]_0^{1/2} = 0$$

$$\left[a_1\left(\frac{(1/2)^2}{2}-\frac{(1/2)^4}{4}\right) + a_2\left(\frac{(1/2)^2}{2}-\frac{(1/2)^6}{6}\right)\right]\left[-6a_1\left(\frac{(1/2)^2}{2}\right) - 20a_2\left(\frac{(1/2)^4}{4}\right) + 500\left(\frac{(1/2)^3}{3}\right)\right] =$$

$$(-0.1093a_1 + 0.127a_2)(-0.75a_1 - 0.3125a_2 + 20.83) = 0$$

$$-0.75a_1 - 0.3125a_2 + 20.83 = 0 \quad — \text{①}$$

Domain (ii)

$$\int_{1/2}^1 \left[a_1(x-x^3) + a_2(x-x^5)\right]\left[-6a_1x - 20a_2x^3 + 500x^2\right] dx = 0$$

$$\left[a_1\left[\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\right] - a_2\left[\left(\frac{x^2}{2}-\frac{x^6}{6}\right)\right]\right]_{1/2}^1\left[-6a_1\left(\frac{x^2}{2}\right) - 20a_2\left(\frac{x^4}{4}\right) + 500\left(\frac{x^3}{3}\right)\right]_{1/2}^1$$

$$\left(-3a_1x^2 - 5a_2x^4 + 166.66x^3\right)_{1/2}^1 = 0$$

$$(-3a_1 + 0.75a_1) + (-5a_2 + 0.3125a_2) + (166.66 - 20.8325) = 0$$

$$-2.25a_1 - 4.6875a_2 + 145.827 = 0$$

$$2.25a_1 + 4.6875a_2 = 145.827 \quad — \text{②}$$

From the equations ① & ②

$a_1 = 18.51$

$a_2 = 22.22$

4. The differential phenomenon of the $\frac{d^2y}{dx^2} + 500x^2 = 0, \; 0 \le x \le 1$

Trial function, $y = a_1(x - x^4)$

Boundary conditions, $y(0) = 0, \; y(1) = 0$

Solution

$y = a_1(x - x^4)$

$y(0) = a_1(0 - 0^4) = 0$

$y(1) = a_1(1 - 1^4) = 0$

$\therefore$ The given boundary condition is satisfied

1. Point Collocation method

$$y = a_1(x - x^4)$$

$$\frac{dy}{dx} = a_1(1 - 4x^3)$$

$$\frac{d^2y}{dx^2} = a_1(-12x^2)$$

Residuals,

$$R = \frac{d^2y}{dx^2} + 500x^2 = 0$$

$$R = -12a_1x^2 + 500x^2 = 0$$

Select the intervals between 0 to 1 $(x = 1/2)$

$$-12a_1\left(\frac{1}{2}\right)^2 + 500\left(\frac{1}{2}\right)^2 = 0$$

$$-3a_1 + 125 = 0$$

$$-3a_1 = -125$$

$$a_1 = 41.66$$

Trial function,

$$y = 41.66(x - x^4)$$

2. Subdomain Collocation method

$$\int_0^1 R \, dx = 0$$

$$\int_0^1 (-12a_1x^2 + 500x^2) \, dx = 0$$

$$-12a_1\left(\frac{x^3}{3}\right)_0^1 + 500\left(\frac{x^3}{3}\right)_0^1 = 0$$

$$-12a_1\left(\frac{1^3}{3}\right)+500\left(\frac{1^3}{3}\right)=0$$

$$-4a_1+166.66=0$$

$$a_1=41.66$$

3. Least square method

$$I=\int_0^1 R^2\,dx=0$$

$$\frac{\partial I}{\partial a_1}=\int_0^1 R\frac{\partial R}{\partial a_1}\,dx=0$$

$$R=-12a_1x^2+500x^2$$

$$\frac{\partial R}{\partial a_1}=-12x^2$$

$$\int_0^1(-12a_1x^2+500x^2)(-12x^2)=0$$

$$\int_0^1\left[144a_1x^4+(-6000x^4)\right]=0$$

$$144a_1\left(\frac{x^5}{5}\right)_0^1-6000\left(\frac{x^5}{5}\right)_0^1=0$$

$$144a_1\left(\frac{1^5}{5}\right)-6000\left(\frac{1^5}{5}\right)=0$$

$$28.8a_1-1200=0$$

$$a_1=41.66$$

4. Galerkins method

$$\int_0^1 W_i\times R\,dx=0$$

The residuals $=-12a_1x^2+500x^2$

$$W_i=(x-x^4)$$

$$\int_0^1(x-x^4)(-12a_1x^2+500x^2)\,dx=0$$

$$\left(\frac{x^2}{2}-\frac{x^5}{5}\right)_0^1\left[-12a_1\left(\frac{x^3}{3}\right)_0^1+500\left(\frac{x^3}{3}\right)_0^1\right]=0$$

$$\left(\frac{1^2}{2}-\frac{1^5}{5}\right)\left[-12a_1\left(\frac{1}{3}\right)+500\left(\frac{1}{3}\right)\right]=0$$

$$0.3(-4a_1+166.66)=0$$

$$a_1=41.66$$

5. Find the Eigen values of $A = \begin{bmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{bmatrix}$

Solution

The characteristic equation,

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \quad \text{--- ①}$$

a_1 = Sum of the leading diagonal elements

$= 4 + 10 - 13$

$a_1 = 1$

a_2 = Sum of the minor of the leading diagonal elements

$$a_2 = \begin{vmatrix} 10 & 4 \\ -30 & -13 \end{vmatrix} + \begin{vmatrix} 4 & -10 \\ 6 & -13 \end{vmatrix} + \begin{vmatrix} 4 & -20 \\ -2 & 10 \end{vmatrix}$$

$= \overset{-10}{\cancel{250}} + 8 + 0$

$a_2 = -2$

$$a_3 = |A| = \begin{vmatrix} 4 & -20 & -10 \\ -2 & 10 & 4 \\ 6 & -30 & -13 \end{vmatrix}$$

$$= 4(-130-(-120)) + 20(26-24) - 10(60-60)$$

$a_3 = 0$

Sub a_1, a_2, a_3 values in ①

$\lambda^3 - (1)\lambda^2 - 2\lambda - 0 = 0$

$\lambda^3 - \lambda^2 - 2\lambda = 0$

$\lambda(\lambda^2 - \lambda - 2) = 0$

$\lambda = 0, \quad \lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2, \lambda = -1$

The eigen values are $(0, 2, -1)$

6. Find $|A|$ and eigen values of $A = \begin{bmatrix} -1 & 3 & -2 \\ 2 & -4 & 2 \\ 0 & 4 & 1 \end{bmatrix}$

Solution

The characteristic equation,

$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$ — ①

a_1 = Sum of the leading diagonal elements

$a_1 = -1-4+1 = -4$

a_2 = Sum of minors of leading diagonal elements

$a_2 = (-4-8) + (-1+0) + (4-6)$

$= -12-1-2$

$a_2 = -15$

$a_3 = |A| = \begin{vmatrix} -1 & 3 & -2 \\ 2 & -4 & 2 \\ 0 & 4 & 1 \end{vmatrix}$

$= -1(-4-8) - 3(2-0) - 2(8+0)$

$= 12-6-16$

$a_3 = -10$

Sub a_1, a_2, a_3 values in ①

$\lambda^3 - (-4)\lambda^2 + (-15)\lambda + 10 = 0$

$\lambda^3 + 4\lambda^2 - 15\lambda + 10 = 0$

$\lambda = 1.5, -6.5, 1$

7. Find the cofactors of matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

Solution

$a_{11} = 12$

$a_{12} = 3-0 = 3$

$a_{13} = 0-8 = -8$

$a_{21} = 3-0 = 3$

$a_{22} = 9-4 = 5$

$a_{23} = 0-2 = -2$

$a_{31} = 0-8 = -8$

$a_{32} = 0-2 = -2$

$a_{33} = 12-1 = 11$

3. Use Gauss elimination method

i) $3x+y-z=3$

$2x-8y+z=-5$

$x-2y+9z=8$

Solution

$$= \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 9 & 8 \\ 2 & -8 & 1 & -5 \\ 3 & 1 & -1 & 3 \end{array}\right] \quad R_3 \leftrightarrow R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 9 & 8 \\ 0 & -26 & 5 & -21 \\ 0 & 7 & -28 & -21 \end{array}\right] \begin{array}{l} 3R_2-2R_3 \\ R_3-3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 9 & 8 \\ 0 & -26 & 5 & -21 \\ 0 & 0 & -693 & -693 \end{array}\right] \quad 26R_3+7R_2$$

$-693z = -693$

$z=1$

$-26y+5z=-21$

$y=1$

$x-2y+9z=8$

$x-2+9=8$

$x=1$

$(x,y,z)=(1,1,1)$

ii) $2x_1+4x_2+2x_3=15$

$2x_1+x_2+2x_3=-5$

$4x_1+x_2-2x_3=0$

Solution

$$= \left[\begin{array}{ccc|c} 2 & 4 & 2 & 15 \\ 2 & 1 & 2 & -5 \\ 4 & 1 & -2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7.5 \\ 0 & 1 & 6 & -10 \\ 4 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} R_1/2 \\ 2R_2 - R_3 \\ \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7.5 \\ 0 & 1 & 6 & -10 \\ 0 & -7 & -6 & -30 \end{array}\right] R_3 - 4R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7.5 \\ 0 & 1 & 6 & -10 \\ 0 & 0 & 36 & -100 \end{array}\right] R_3 + 7R_2$$

$$36Z = -100$$
$$Z = -2.7$$
$$y + 6Z = -10$$
$$y + 6(-2.7) = -10$$
$$y = 6.2$$
$$x + 2y + Z = 7.5$$
$$x + 2(6.2) - 2.7 = 0$$
$$x + 12.4 - 2.7 = 0$$
$$x = -9.7$$

Pre Processing ← [Discretization
↓
No. of nodes & elements]
↓
Analysis ← [Selection of displacement function
↓
Define the material behaviour
↓
Derivation of element stiffness matrix & equations.
↓
Assemble the element eqns
↓
Applying boundary conditions
↓
Solution of unknown displacements
↓
Computation of element stresses & strains]
↓
Post Processing ← Interpret the results

RAYLEIGH - RITZ METHOD (Variation Method)

For continuous system,

$$\text{Potential energy } (\pi) = \int_{x_1}^{x_2} f(y, y', y'')\, dx$$

Total potential energy of the structure is,

$$\pi = \left\{\begin{array}{c}\text{Internal potential} \\ \text{energy}\end{array}\right\} - \left\{\begin{array}{c}\text{External potential} \\ \text{energy}\end{array}\right\}$$

= Strain energy - Work done by external forces

$$\boxed{\pi = U - H}$$

The polynomial (or) Trigonometric series with undetermined constants,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

(or)

$$* \; y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l}$$

The constant, $a_0, a_1, a_2 \rightarrow$ Unknown Ritz parameters

The strain energy due to bending moment,

$$* \; U = \frac{EI}{2} \int_{x_1}^{x_2} \left(\frac{d^2y}{dx^2} \right)^2 dx$$

Work done by external force,

$$* \; H = \int_{x_1}^{x_2} w \times y \times dx$$

$y \rightarrow$ Undetermined polynomial constants.

Formulas used

$* \; (a+b)^2 = a^2 + b^2 + 2ab$

$* \; \sin^2 x = \frac{1 - \cos 2x}{2}$

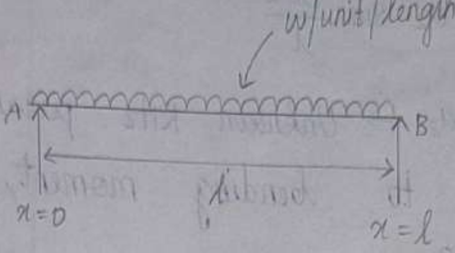
$* \; \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$

⇒ Rayleigh - Ritz method is an integral approach method which is useful for solving complex structural problems encounted in Finite Element Analysis.

⇒ Two conditions must be fulfilled by approximation functions such as

1. It should satisfy the geometry boundary conditions.
2. The function must have atleast one Ritz parameter.

1. A Simply Supported Beam subjected to Uniformly Distributed load over entire span length. Determine the Bending Moment and deflection at midspan by using Rayleigh Ritz method and compare with exact solution.



$\frac{d}{dx}$ $\sin\theta\, dx = \cos\theta$

$\cos\theta\, dx = -\sin\theta$

$\int$ $\int \sin\theta\, dx = -\cos\theta$

$\int \cos\theta\, dx = \sin\theta$

Solution

For Simply Supported Beam, The fourier series

$$y = \sum_{n=1,3}^{\alpha} a \sin\frac{n\pi x}{l}$$

The function,

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

The potential energy of the Beam,

$$\pi = U - H$$

$\downarrow$ $\hookrightarrow$ Workdone by external force

Strain energy

Strain energy, U of the beam due to bending,

$$U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

$$\frac{dy}{dx} = a_1 \cos\frac{\pi x}{l}\left(\frac{\pi}{l}\right) + a_2 \cos\frac{3\pi x}{l}\left(\frac{3\pi}{l}\right)$$

$$= \frac{a_1\pi}{l}\cos\frac{\pi x}{l} + \frac{3a_2\pi}{l}\left(\frac{3\pi l}{l}\right)\cos$$

$$\frac{d^2y}{dx^2} = -\frac{a_1\pi}{l}\sin\frac{\pi x}{l}\left(\frac{\pi}{l}\right) - \frac{3a_2\pi}{l}\sin\left(\frac{3\pi x}{l}\right)\left(\frac{3\pi}{l}\right)$$

$$= -\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}$$

Strain energy,

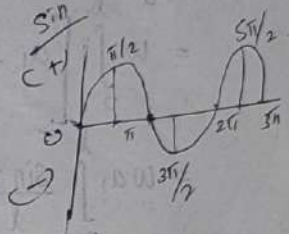
$$U = \frac{EI}{2}\int_0^l \left[-\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - \frac{9a_2\pi^2}{l^2}\sin\frac{3\pi x}{l^2}\right]^2 dx$$

$$= \frac{EI}{2}\times\frac{\pi^4}{l^4}\int_0^l \left[a_1\frac{\sin \pi x}{l} + 9a_2\frac{\sin 3\pi x}{l}\right]^2 dx$$

$$U = \frac{EI}{2}\times\frac{\pi^4}{l^4}\int_0^l \left[a_1^2\frac{\sin^2\pi x}{l} + 81a_2^2\sin^2\frac{3\pi x}{l} + 18a_1a_2\frac{\sin\pi x}{l}\cdot\sin\frac{3\pi x}{l}\right]dx$$

I $$\int_0^l \left(a_1^2\sin^2\frac{\pi x}{l}\right)dx = a_1^2\int_0^l \frac{1}{2}\left(1-\cos\frac{2\pi x}{l}\right)dx$$

$$= \frac{a_1^2}{2}\int_0^l dx - \frac{a_1^2}{2}\int_0^l \cos\frac{2\pi x}{l}\,dx$$



$$= \frac{a_1^2}{2}\left[(x)_0^l - \left(\frac{\sin\frac{2\pi x}{l}}{2\pi/l}\right)_0^l\right]$$

$$= \frac{a_1^2}{2}\left[l - \left(\frac{\sin\frac{2\pi l}{l}}{2\pi/l}\right)\right]$$

$$= \frac{a_1^2 l}{2}$$

II $$\int_0^l 81a_2^2\sin^2\left(\frac{3\pi x}{l}\right)dx = 81a_2^2\int_0^l\left[\frac{1-\cos 2\left(\frac{3\pi x}{l}\right)}{2}\right]dx$$

$$= \frac{81}{2}a_2^2\int_0^l\left[1-\cos\left(\frac{6\pi x}{l}\right)\right]dx$$

$$= \frac{81}{2}a_2^2\left[(x)_0^l - \left(\frac{\sin\left(\frac{6\pi x}{l}\right)}{6\pi/l}\right)\right]_0^l$$

$$= \frac{81a^2 l}{2}$$

III $$\int_0^l 18a_1a_2\frac{\sin\pi x}{l}\cdot\sin\left(\frac{3\pi x}{l}\right)dx = 18a_1a_2\int_0^l\left[\frac{\sin\pi x}{l}\cdot\sin\frac{3\pi x}{l}\right]dx$$

SinA · SinB

$$= \frac{18}{2}a_1a_2\int_0^l\left[\cos\left(\frac{\pi x}{l}-\frac{3\pi x}{l}\right) - \cos\left(\frac{\pi x}{l}+\frac{3\pi x}{l}\right)\right]dx$$

$$= \frac{18}{2}a_1a_2\int_0^l\left[\cos\left(\frac{2\pi x}{l}\right) - \cos\left(\frac{4\pi x}{l}\right)\right]dx$$

$\cos(-\theta) = \cos\theta$

$$= \frac{18}{2}a_1a_2\int_0^l\cos\left(\frac{2\pi x}{l}\right)dx - \frac{18}{2}a_1a_2\int_0^l\cos\left(\frac{4\pi x}{l}\right)dx$$

$$= \frac{18}{2} a_1 a_2 \left[\sin\left(\frac{2\pi x}{l}\right) \times \frac{l}{2\pi} \right]_0^l - \frac{18}{2} a_1 a_2 \left[\sin\left(\frac{4\pi x}{l}\right) \times \frac{l}{2\pi} \right]_0^l$$

$$= 0$$

$$U = \frac{EI}{2} \times \frac{\pi^4}{l^4} \left[\frac{a_1^2 l}{2} + \frac{81 a_2^2 l}{2} + 0 \right]$$

$$= \frac{EI}{2} \times \frac{\pi^4}{l^4} \times \frac{l}{2} \left[a_1^2 + 81 a_2^2 \right]$$

$$U = \frac{EI\pi^4}{4l^3} \left[a_1^2 + 81 a_2^2 \right]$$

$$H = \int_0^l w \cdot y \, dx$$

$$= w \int_0^l \left[a_1 \frac{\sin \pi x}{l} + a_2 \frac{\sin 3\pi x}{l} \right] dx$$

$$= w a_1 \int_0^l \frac{\sin \pi x}{l} dx + w a_2 \int_0^l \frac{\sin 3\pi x}{l} dx$$

$$= w a_1 \left[-\frac{\cos \pi x}{l} \times \frac{l}{\pi} \right]_0^l + w a_2 \left[-\cos \frac{3\pi x}{l} \times \frac{l}{3\pi} \right]_0^l$$

$$= w a_1 \left[\frac{l}{\pi} + \frac{l}{\pi} \right] + w \cdot a_2 \left[\frac{l}{3\pi} + \frac{l}{3\pi} \right]$$

$$= \frac{2 w a_1 l}{\pi} + \frac{2 w a_2 l}{3\pi}$$

$$= \frac{6 w a_1 l + 2 w \cdot a_2 l}{3\pi}$$

$$H = \frac{2 l \cdot w}{3\pi} \left[3a_1 + a_2 \right]$$

$$\pi = U - H$$

$$= \frac{EI\pi^4}{4l^3} \left[a_1^2 + 81 a_2^2 \right] - \frac{2lw}{3\pi} \left[3a_1 + a_2 \right]$$

$$\frac{\partial \pi}{\partial a_1} = \frac{EI\pi^4}{4l^3} 2a_1 - \frac{2lw}{\pi}$$

Calculating a_1, $\frac{\partial \pi}{\partial a_1} = 0$

$$0 = \frac{EI\pi^4 a_1}{2l^3} - \frac{2lw}{\pi}$$

$$\frac{EI\pi^4 a_1}{2l^3} = \frac{2lw}{\pi}$$

$$a_1 = \frac{2lw}{\pi} \times \frac{2l^3}{EI\pi^4}$$

$$a_1 = \frac{4l^4 w}{E\pi^5 I}$$

$$\frac{\partial \pi}{\partial a_2} = 162\, a_2 \frac{EI\pi^4}{4l^3} - \frac{2l.w}{3\pi} = 0$$

$$40.5\, a_2 \frac{EI\pi^4}{l^3} = \frac{2\,l.w}{3\pi}$$

$$a_2 = \frac{2.l.w}{3\pi} \times \frac{l^3}{E.I\,\pi^4 (40.5)}$$

$$a_2 = \frac{0.016\, l^4 w}{E.I.\pi^5}$$

Sub a_1 & a_2 values,

$$y = \left(\frac{4l^4 w}{E\pi^5 I}\right) \sin\left(\frac{\pi x}{l}\right) + \left(\frac{0.016\, l^4 w}{EI\pi^5}\right) \sin \frac{3\pi x}{l}$$

$$y = \frac{l^4 w}{E\pi^5 I}\left[4 \sin\frac{\pi x}{l} + 0.016 \sin \frac{3\pi x}{l}\right]$$

For maximum deflection, $x = \frac{l}{2}$ (Midpoint)

$$y_{max} = \frac{l^4 w}{EI\pi^5}\left[4 \sin\left(\frac{\pi}{2}\right) + 0.016 \sin\left(\frac{3\pi}{2}\right)\right]$$

$$= \frac{l^4 w}{EI\pi^5}[4 - 0.016]$$

$$= \frac{l^4 w}{EI}\left[\frac{3.984}{\pi^5}\right]$$

$$y_{max} = 0.0130 \frac{l^4 w}{EI}$$

Bending moment at midspan,

W.K.T

Bending moment, $\boxed{M = EI \frac{d^2y}{dx^2}}$

$$\frac{d^2y}{dx^2} = \left[-\frac{a_1\pi^2}{l^2} \sin\left(\frac{\pi x}{l}\right) - \frac{9a_2}{l^2} \sin\left(\frac{3\pi x}{l}\right)\right]$$

Sub a_1 & a_2

$$= \left[\frac{-4l^4 w}{EI\pi^5} \times \frac{\pi^2}{l^2} \sin\left(\frac{\pi x}{l}\right)\right] - \left[\frac{9 \times 0.016\, l^4 w}{EI\pi^5} \times \frac{\pi^2}{l^2} \sin\left(\frac{3\pi x}{l}\right)\right]$$

$$= \left[\frac{-4l^2 w}{EI\pi^3}\sin\left(\frac{\pi x}{l}\right) - \frac{0.144\, l^2 w}{EI\pi^3}\sin\left(\frac{3\pi x}{l}\right)\right]$$

$$\frac{d^2y}{dx^2} = \frac{-l^2 w}{EI\pi^3}\left[4\sin\left(\frac{\pi x}{l}\right) + 0.144\sin\left(\frac{3\pi x}{l}\right)\right]$$

$$M = EI \times \frac{d^2y}{dx^2}$$

$$= EI \times \frac{-l^2 w}{EI\pi^3}\left[4\sin\left(\frac{\pi x}{l}\right) + 0.144\sin\left(\frac{3\pi x}{l}\right)\right]$$

$$M = \frac{-l^2 w}{\pi^3}\left[4\sin\left(\frac{\pi x}{l}\right) + 0.144\sin\left(\frac{3\pi x}{l}\right)\right]$$

Max. Bending moment at $x = \frac{l}{2}$ (Midspan)

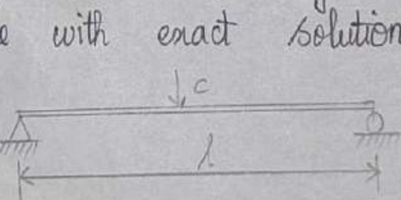
$$= \frac{-l^2 w}{\pi^3}\left[4\sin\left(\frac{\pi}{2}\right) \pm 0.144\sin\left(\frac{3\pi}{2}\right)\right]$$

$$= \frac{-l^2 w}{\pi^3}\left[4 - 0.144\right]$$

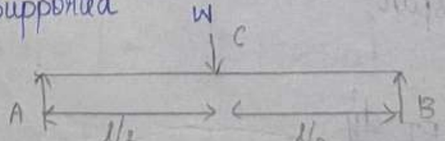
$$= -l^2 w\left[\frac{3.856}{\pi^3}\right]$$

$$M = -0.124\, l^2 w$$

2. Beam AB of span length, l Simply supported at both the ends and carrying a concentrate load at Centre 'C'. Determine the deflection at the midspan of the beam by using Rayleigh-Ritz method and compare with exact solution.



Simply supported



Soln

'U' calculated in previous sum

$$U = \frac{EI\pi^4}{4l^3}\left[a_1^2 + 81a_2^2\right]$$

External force → Point load

$H = W \cdot y_{max}$

$y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l}$

Sub $x = l/2$

$y_{max} = a_1 \sin \frac{\pi (l/2)}{l} + a_2 \frac{\sin 3\pi (l/2)}{l}$

$= a_1 \sin\left(\frac{\pi}{2}\right) + a_2 \sin\left(\frac{3\pi}{2}\right)$

$y_{max} = a_1 - a_2$

$H = W(a_1 - a_2)$

$\pi = U - H$

$= \frac{EI\pi^4}{4l^3}\left[a_1^2 + 81 a_2^2\right] - W(a_1 - a_2)$

$\frac{\partial \pi}{\partial a_1} = 0$

$\frac{EI\pi^4}{4l^3}[2a_1] - W(1-0) = 0$

$\frac{EI\pi^4}{4l^3}[2a_1] = W$

$a_1 = \frac{4Wl^3}{2EI\pi^4}$

$= \frac{2Wl^3}{EI\pi^4}$

Similarly,

$\frac{\partial \pi}{\partial a_2} = 0$

$a_2 = \frac{-2Wl^3}{81 EI\pi^4}$

Sub a_1 & a_2 in y_{max}

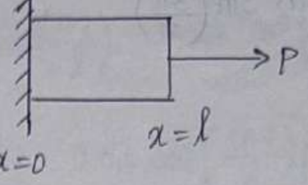
$y_{max} = \frac{2Wl^3}{EI\pi^4} + \frac{2Wl^3}{81EI\pi^4}$

$= \frac{2Wl^3}{EI\pi^4}\left[1 + \frac{1}{81}\right]$

$= \frac{2Wl^3}{EI}\left[\frac{1.012}{\pi^4}\right]$

$y_{max} = 0.0207 \frac{Wl^3}{EI}$

3. A bar of uniform cross section is clamped at one end and left free at the other end and it is subjected to a uniform axial load, P as shown in fig. Calculate the displacement and stress in a bar by using two terms polynomial & three terms polynomial and compare with exact solution.



Solution

Case (i) Two polynomial

$u = a_0 + a_1 x$

Boundary conditions $u = 0, x = 0$

$0 = a_0 + a_1(0)$

$a_0 = 0$

Sub $a_0 = 0$ in u

$u = 0 + a_1 x$

Diff 'u' w.r to 'x'

$$\boxed{\frac{du}{dx} = a_1}$$

Total potential energy,

$\pi = U - H$

where, $U \rightarrow$ Strain energy

$H \rightarrow$ Workdone by external force

Strain energy,

$$U = \frac{EA}{2}\int_0^l \left(\frac{du}{dx}\right)^2 dx$$

$$= \frac{EA}{2}\int_0^l (a_1)^2 dx$$

$$= \frac{EA a_1^2}{2}(x)_0^l$$

$$U = \frac{EA a_1^2 l}{2}$$

Work done by external force,

$$H = \int_0^l P \cdot dx$$

$$= \int_0^l \rho U A \, dx$$

$$= \rho A \int_0^l U \, dx$$

$$= \rho A \int_0^l a_1 x \, dx$$

$$= \rho A a_1 \left(\frac{x^2}{2}\right)_0^l$$

$$H = \frac{\rho A a_1 l^2}{2}$$

The eqn, $\pi = U - H$

$$= \frac{EAa_1^2 l}{2} - \frac{\rho A a_1 l^2}{2}$$

$$\pi = \frac{A \cancel{a_1} l}{\cancel{2}} \left[E A_1 - / \rho \cancel{l} \right]$$

$$\frac{\partial \pi}{a_1} = 0$$

$$\frac{2EA_1 l}{2} - \frac{\rho A l^2}{2} = 0$$

$$2EA_1 l \cancel{a_1} = \rho A l^2$$

$$a_1 = \frac{\rho A l^2}{2EA_1 l}$$

$$a_1 = \frac{\rho l}{2E}$$

Sub a_1 in U

$$U = a_1 x = \frac{\rho l x}{2E}$$

Sub $x = l$ in U

$$U = U_1 = \frac{\rho l (l)}{2E} = \frac{\rho l^2}{2E}$$

Extension of the bar,

$$\delta u = U_1 - U_0$$

$$= \frac{\rho l^2}{2E} - 0$$

$$\delta u = \frac{\rho l^2}{2E}$$

Stress of the bar $= E\dfrac{du}{dx}$

$= E \times \dfrac{\rho l}{2E}$

$= \dfrac{\rho l}{2}$

Case (ii) Three polynomial

$U = a_0 + a_1 x + a_2 x^2$

Boundary conditions, $x = 0$, $u = 0$

$0 = a_0 + a_1(0) + a_2(0)^2$

$a_0 = 0$

Sub $a_0 = 0$ in U

$U = a_1 x + a_2 x^2$

Diff U w.r to x

$\dfrac{du}{dx} = a_1 + 2a_2 x$

Total potential energy,

$\pi = U - H$

Strain energy,

$$U = \frac{EA}{2}\int_0^l \left(\frac{du}{dx}\right)^2 dx$$

$$= \frac{EA}{2}\int_0^l (a_1 + 2a_2 x)^2 dx$$

$$= \frac{EA}{2}\int_0^l (a_1^2 + 4a_2^2 x^2 + 4a_1 a_2 x)\, dx$$

$$= \frac{EA}{2}\, a_1^2 (x)_0^l + 4a_2^2\left(\frac{x^3}{3}\right)_0^l + 4a_1 a_2\left(\frac{x^2}{2}\right)_0^l$$

$$= \frac{EA}{2}\, a_1^2 l + 4a_2^2 \frac{l^3}{3} + 2a_1 a_2 l^2$$

$$U = \frac{EA}{6}\left(3a_1^2 l + 4a_2^2 l^3 + 6a_1 a_2 l^2\right)$$

Work done by external force,

$$H = \int_0^l P.dx$$

$$= \int_0^l \rho U A\, dx$$

$$= \rho A \int_0^l u \, dx$$

$$= \rho A \int_0^l a_1 x + a_2 x^2 \, dx$$

$$= \rho A \, a_1 \left(\frac{x^2}{2}\right)_0^l + a_2 \left(\frac{x^3}{3}\right)_0^l$$

$$= \rho A \, a_1 \left(\frac{l^2}{2}\right) + a_2 \left(\frac{l^3}{3}\right)$$

$$= \rho A \left(\frac{3a_1 l^2}{2} + \frac{2a_2 l^3}{3}\right)$$

$$H = \frac{\rho A}{6}\left(3a_1 l^2 + 2a_2 l^3\right)$$

The equation,

$$\pi = U - H$$

$$\pi = \frac{EA}{6}\left(3a_1^2 l + 4a_2^2 l^3 + 6a_1 a_2 l^2\right) - \frac{\rho A}{6}\left(3a_1 l^2 + 2a_2 l^3\right)$$

Diff π w.r to a_1

$$\frac{\partial \pi}{\partial a_1} = 0$$

$$\left[\frac{6a_1 l EA}{6} + \frac{6a_2 l^2}{6}\right] - \frac{3l^2 \rho A}{6} = 0$$

$$6EAa_1 l + 6EAa_2 l^2 - 3\rho A l^2 = 0$$

$$6a_1 l + 6a_2 l^2 = \frac{3\rho l^2}{E}$$

$$6l(a_1 + a_2 l) = \frac{3\rho l^2}{E}$$

$$a_1 + a_2 l = \frac{3\rho l^2}{6El}$$

$$a_1 + a_2 l = \frac{\rho l}{2E} \quad \text{--- } (1)$$

Diff π w.r to a_2

$$\frac{\partial \pi}{\partial a_2} = 0$$

$$\left[\frac{EA}{6}\left(8a_2 l^3 + 6a_1 l^2\right)\right] - \left[\frac{\rho A}{6}\left(2l^3\right)\right] = 0$$

$$\frac{8EAa_2 l^3}{6} + \frac{6EAa_1 l^2}{6} - \frac{\rho A 2l^3}{6} = 0$$

$$8EAa_2 l^3 + 6EAa_1 l^2 = 2\rho A l^3$$

$$2l^2 EA(4a_2 l + 3a_1) = 2\rho A l^3$$

$$4a_2 l + 3a_1 = \frac{2\rho A l^3}{2l^2 EA}$$

$$4a_2 l + 3a_1 = \frac{\rho l}{E}$$

$$3a_1 + 4a_2 l = \frac{\rho l}{E} \quad - \text{②}$$

From ① & ②

$$2a_1 + 2a_2 l = \frac{\rho l}{E}$$

$$3a_1 + 4a_2 l = \frac{\rho l}{E} \quad (-)$$

$$-a_1 - 2a_2 l = 0$$

$$a_1 + 2a_2 l = 0$$

$$a_1 + 2a_2 l = 0$$

$$a_1 = -2a_2 l$$

Sub a_1 in ①

$$2(-2a_2 l) + 2a_2 l = \frac{\rho l}{E}$$

$$-4a_2 l + 2a_2 l = \frac{\rho l}{E}$$

$$-2a_2 l = \frac{\rho l}{E}$$

$$a_2 = \frac{-\rho}{2E}$$

Sub a_1 & a_2 in U

$$U = a_1 x + a_2 x^2$$

$$U = -2a_2 l x - \frac{\rho}{2E} x^2$$

Sub $x = l$ in U

$$U = U_1 = -2a_2 l^2 - \frac{\rho}{2E} l^2$$

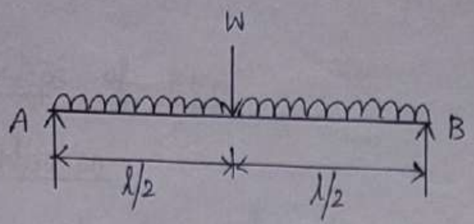
Extension of the bar, $\delta U = U_1 - U_0$

$$\delta U = -2a_2 l^2 - \frac{\rho}{2E} l^2$$

Stress of the bar $= E \cdot \frac{du}{dx}$

$$= E\left(-2a_2 l - \frac{2\rho}{E} x\right)$$

4.



Calculate deflection. Use $H = \left(\int_0^l w \cdot y \, dx\right) + (w \cdot y_{max})\, dx$

Soln

U calculated in 1st sum

$$U = \frac{EI\pi^4}{4l^3}\left[a_1^2 + 81a_2^2\right]$$

$$H = \int_0^l W \cdot y \, dx + (W \cdot y_{max})\, dx$$

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$$

y_{max} occurs at $x = l/2$

$$y_{max} = a_1 \sin\frac{\pi(l/2)}{l} + a_2 \sin\frac{3\pi(l/2)}{l}$$

$$y_{max} = a_1 - a_2$$

$$H = W\int_0^l \left[a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}\right] dx + W\int_0^l (a_1 - a_2)\, dx$$

$$H = Wa_1\left[-\cos\frac{\pi x}{l} \times \frac{l}{\pi}\right]_0^l + Wa_2\left[-\cos\frac{3\pi x}{l} \times \frac{l}{3\pi}\right]_0^l + W(a_1 - a_2)(x)_0^l$$

$$H = Wa_1\left[\frac{l}{\pi} + \frac{l}{\pi}\right] + Wa_2\left[\frac{l}{3\pi} + \frac{l}{3\pi}\right] + (Wa_1 - Wa_2)\, l$$

$$H = \frac{Wa_1 2l}{\pi} + \frac{Wa_2 2l}{3\pi} + Wa_1 l - Wa_2 l$$

$$H = Wl\left[\frac{2a_1}{\pi} + \frac{2a_2}{3\pi} + a_1 - a_2\right]$$

$$H = Wl\left[0.636\, a_1 + 0.212\, a_2 + a_1 - a_2\right]$$

$$H = Wl\left[1.636\, a_1 - 0.788\, a_2\right]$$

$$\pi = U - H$$

$$\pi = \frac{EI\pi^4}{4l^3}\left[a_1^2 + 81a_2^2\right] - Wl\left[1.636\, a_1 - 0.788\, a_2\right]$$

Diff w.r to a_1

$$\frac{\partial \pi}{\partial a_1} = 0$$

$$\frac{EI\pi^4}{4l^3}\left[2a_1\right] - Wl(1.636) = 0$$

$$a_1 = \frac{Wl(1.636) \times 2l^3}{EI\pi^4}$$

$$a_1 = \frac{0.03\,Wl^4}{EI}$$

Diff w.r to a_2

$$\frac{\partial \pi}{\partial a_2} = 0$$

$$162a_2 \frac{EI\pi^4}{4l^3} + Wl(0.788) = 0$$

$$a_2 = \frac{-Wl(0.788) \times 4l^3}{162\,EI\pi^4}$$

$$a_2 = \frac{-3.152\,Wl^4}{15780.2\,EI}$$

$$y_{max} = a_1 - a_2$$

$$= \frac{0.03\,Wl^4}{EI} - \frac{0.00019\,Wl^4}{EI}$$

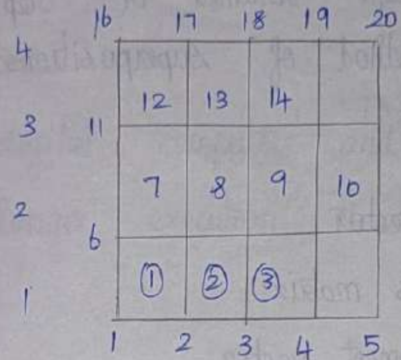
$$y_{max} = \frac{0.029\,Wl^4}{EI}$$

General steps of FEA

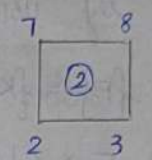
1. Discridisation

The art of subdividing a structure into a convenient no. of smaller elements is known as discridization

2. Numbering of Nodes & elements



Longer side



Max node : 8
Min node : 2
Difference : 6

3. Selection of a displacement function (or) interpolation function

⇒ It involves choosing the displacement function within each element.

⇒ Polynomial of linear, Quadratic & cubic form are frequently used as displacement functions because they are simple to work for finite element formation.

4. Define the material behaviour by using strain displacement and stress strain relationship matrix

Strain displacement and Stress, Strain matrix relationship are necessary to deriving the equations for Finite Element

$$\sigma = E\frac{du}{dx}$$

5. Derivation of Element Stiffness matrix and equations

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

The Finite Element eqn is in Matrix form,

$$[F_e] = \{K^e\} \times \{u^e\}$$

Force vector ↑ — Stiffness matrix ↑ — Displacement vector ↑

6. <u>Assemble the element equations to obtain the Global/Total equations</u>

The individual element eqn obtained in Step 5 are added together by using a method of superpositions

$$\{F\} = \{K\} \times \{u\}$$

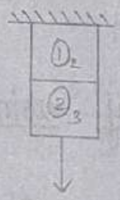
$F \rightarrow$ Global force vector

$K \rightarrow$ Global stiffness matrix

$u \rightarrow$ Global displacement vector

7. <u>Applying boundary conditions</u>

The Global stiffness matrix, K is a singular matrix because its determinent is equal to zero. In order to remove the singularily problems, certain boundary conditions are applied.



8. <u>Solution for the unknown displacement</u>

By using Gauss elimination method, Calculate the unknown displacement functions.

9. <u>Computation of the elements strains and stresses from the nodal displacement function</u>

$$Strain(e) = \frac{du}{dx}$$

$$Stress\,(\sigma) = E \times e$$
$$= E \times \frac{du}{dx}$$

10. Data interpretation

Interpret the results or post process

Advantages of FEA

* FEA/FEM can handle irregular geometry in convenient manner.
* Handles general load conditions without any difficulties.
* All the various types of boundary conditions are handled

Disadvantages of FEA

* It requires digital computers and fairly extensive software.
* It requires longer execution time compare to other methods

Applications

* FEA can be used to analyse both structural and non-structural problems.
* In structural problems, displacement at each node point is applied.
* By using displacement solution, stress & strain can be calculated.
* In non-structural problems, temperature or fluid pressure at each nodal point can be ~~aq~~ calculated.
* By using

UNIT-2

ONE DIMENSIONAL PROBLEMS

One dimensional elements

1. Bar element
2. Truss element
3. Beam element

Natural Co-Ordinates

$$\oint (L_1)^{\alpha}(L_2)^{\beta}(L_3)^{\gamma}\, dA = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!} \times 2A$$

$$\int_{x_1}^{x_2} L_1^{\alpha} L_2^{\beta}\, dx = \frac{\alpha!\beta!}{(\alpha+\beta+1)!} \times l$$

Two nodal truss element

Displacement function,

$$U = N_1 u_1 + N_2 u_2$$

where,

$$N_1 = \frac{l-x}{l}, \quad N_2 = \frac{x}{l}$$

Strain, $e = [B]\{u^*\}$

$[B]$ - Strain displacement matrix

$\{u^*\}$ - Degree of freedom

Stress $(\sigma) = E \times e$

Stiffness matrix,

$$[K] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Strain energy,

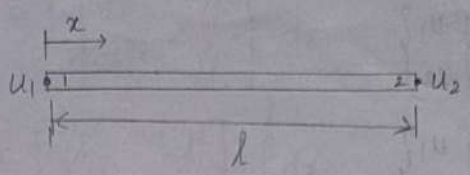
$$U = \frac{1}{2} \times \{u^*\}^T \times [K] \times \{u^*\}$$

$$\delta L = \frac{PL}{AE}$$

Stiffness matrix,

$$[K] = \int_V [B]^T [D] [B]\, dv$$

Derivation of the displacement function 'u' and shape function 'n' for One-Dimensional linear bar element based on Global Co-ordinate approach.



The element has two DOF, it will have two generalised co-ordinate approach,

$$u = a_0 + a_1 x \quad \text{--- } ①$$

$a_0, a_1 \rightarrow$ Global (or) Generalised co-ordinates

$$u = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \quad \text{--- } ②$$

At node 1, $u = u_1$, $x = 0$

At note 2, $u = u_2$, $x = l$

Sub the above values in ①

$$u_1 = a_0 \quad \text{--- } ③$$

$$u_2 = a_0 + a_1 l \quad \text{--- } ④$$

Arranging the eqn ③ & ④ in matrix form,

$$\underbrace{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{u^*} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}}_{C} \underbrace{\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}}_{A}$$

where, $u^* \rightarrow$ DOF

$C \rightarrow$ Connectivity matrix

$A \rightarrow$ Generalised (or) Global coordinate matrix

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l-0} \begin{bmatrix} l & -0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Note: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \dfrac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{--- (5)}$$

Sub eqn (5) in (2)

$$u = \begin{bmatrix} 1 & x \end{bmatrix} \frac{1}{l}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

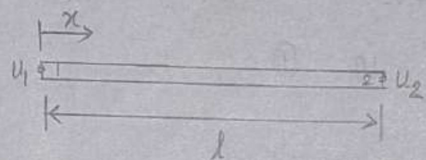
$$= \frac{1}{l}\begin{bmatrix} 1 & x \end{bmatrix}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = \frac{1}{l}\begin{bmatrix} l-x & x+0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{l-x}{l}, & \frac{x}{l} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Derivation of stiffness matrix for one dimensional bar element



Stiffness matrix,

$$[K] = \int_V [B]^T [D] [B] \, dv$$

where, $[B]$ Strain displacement matrix

$[D]$ Stress strain relationship matrix

Displacement function, $u = N_1 u_1 + N_2 u_2$

where, $N_1 = \frac{l-x}{l}$, $N_2 = \frac{x}{l}$

Strain displacement function,

$$[B] = \begin{bmatrix} \frac{d(N_1)}{dx}, & \frac{d(N_2)}{dx} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{l}, & \frac{1}{l} \end{bmatrix}$$

$$[B]^T = \begin{bmatrix} -1/l \\ 1/l \end{bmatrix}$$

$$[K] = \int_0^l \begin{bmatrix} -1/l \\ 1/l \end{bmatrix} \times E \times \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dv$$

$$[K] = \int_0^l \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} E\, dv$$

$$dv = A\, dx$$

$$[K] = AE \int_0^l \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} dx$$

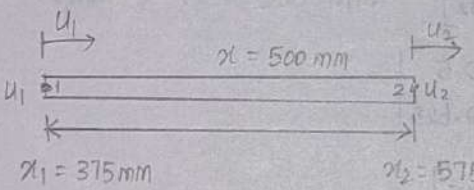
$$= AE \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} \int_0^l dx$$

$$= AE \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} (x)_0^l$$

$$= AE \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} \times l$$

$$[K] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

1.



$u_1 = 0.5$ mm
$u_2 = 0.625$ mm
$A = 750$ mm^2
$E = 2\times10^5$ N/mm^2.

Consider a bar element as shown in fig. The cross sectional area of the bar is 750 mm^2 & Youngs modulus is 2×10^5 N/mm^2. If $u_1 = 0.5$ mm, $u_2 = 0.625$, Calculate (i) Displacement at point P(G) (ii) Strain (e) (iii) Stress (σ) iv) Element stiffness matrix (K) v) Strain energy (U)

Given

$A = 750$ mm^2

$E = 2\times10^5$ N/mm^2

$u_1 = 0.5$ mm

$u_2 = 0.625$ mm

$x_1 = 375$ mm

$x_2 = 575$ mm

$x = 500$ mm

Solution

Actual length of the bar, $l = 575 - 375$

$l = 200$ mm

Distance between point 1 and P is

$x = 500 - 375$

$x = 125$ mm

W.K.T,

$$u = N_1 u_1 + N_2 u_2$$

$$= \left(\frac{l-x}{l}\right) u_1 + \left(\frac{x}{l}\right) u_2$$

$$= \left(\frac{200-125}{200}\right) 0.5 + \left(\frac{125}{200}\right) \times 0.625$$

$$u = 0.578 \text{ mm}$$

Strain $(e) = [B]\{u^*\}$

$$= \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{200} & \frac{1}{200} \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.625 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-0.5}{200} + \frac{0.625}{200} \end{bmatrix}$$

$$e = 6.25 \times 10^{-4}$$

Stress $(\sigma) = E \times e$

$$= 2 \times 10^5 \times 6.25 \times 10^{-4}$$

$$\sigma = 125 \text{ N/mm}^2$$

Stiffness matrix $[k] = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \frac{750 \times 2 \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k] = 750{,}000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Strain energy, $U = \frac{1}{2} \{u^*\}^T \times [k] \times \{u^*\}$

$$= \frac{1}{2} \begin{bmatrix} 0.5 & 0.625 \end{bmatrix} \times 750{,}000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 0.5 \\ 0.625 \end{bmatrix}$$

$$= \frac{750{,}000}{2}\begin{bmatrix}0.5 & 0.625\end{bmatrix}\begin{bmatrix}0.5-0.625\\-0.5+0.625\end{bmatrix}$$

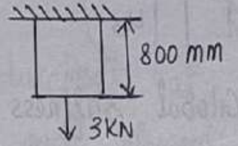
$$= \frac{750{,}000}{2}\begin{bmatrix}0.5 & 0.625\end{bmatrix}\begin{bmatrix}-0.125\\0.125\end{bmatrix}$$

$$= \frac{750{,}000}{2}\left[(0.5\times -0.125)+(0.625\times 0.125)\right]$$

$$= \frac{750{,}000}{2}\left[(-0.0625)+(0.0781)\right]$$

$$\boxed{U = 5859.37\ \text{N-mm}}$$

2. A steel bar of length 800 mm is subjected to an axial load of 3 KN as shown in fig.



Find the elongation of the bar neglecting the self weight.
Take $E = 2\times10^5\ N/mm^2$, $A = 300\ mm^2$

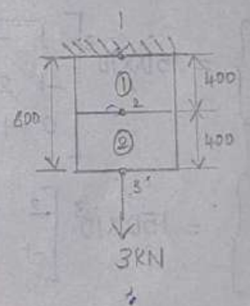
Solution

$l = 800$ mm

$E = 2\times10^5\ N/mm^2$

$A = 300\ mm^2$

$P = 3\times10^3$ N



For node 1 & 2 [element 1];

Finite element equation,

$$\begin{Bmatrix}F_1\\F_2\end{Bmatrix} = \frac{A_1E_1}{l_1}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}$$

$$= \frac{2\times10^5\times300}{400}\begin{array}{c}\begin{array}{cc}a_{11} & a_{12}\end{array}\\\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\\\begin{array}{cc}a_{21} & a_{22}\end{array}\end{array}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}$$

$$\begin{Bmatrix}F_1\\F_2\end{Bmatrix} = 150\times10^3\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} \quad \text{--- } ①$$

For node 2 & 3 [Element 2],

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{A_2 E_2}{l_2} \begin{array}{cc} a_{22} & a_{23} \\ \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{array}{cc} a_{32} & a_{33} \end{array}$$

$$= \frac{300 \times 2 \times 10^5}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = 150 \times 10^3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Assemble the Finite Element equation ① & ②

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 150 \times 10^3 \begin{bmatrix} \overset{a_{11}}{1} & \overset{a_{12}}{-1} & \overset{a_{13}}{0} \\ \overset{a_{21}}{-1} & \overset{a_{22}}{(1+1)} & \overset{a_{23}}{-1} \\ \overset{a_{31}}{0} & \overset{a_{32}}{-1} & \overset{a_{33}}{1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

↑ Global stiffness matrix

Applying boundary conditions,

$u_1 = 0$	$f_1 = 0$
$u_2 =$	$f_2 = 0$
$u_3 =$	$f_3 = 3 \times 10^3$ N

$$\begin{Bmatrix} 0 \\ 0 \\ 3 \times 10^3 \end{Bmatrix} = 150 \times 10^3 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 3 \times 10^3 \end{Bmatrix} = 150 \times 10^3 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$150 \times 10^3 [2u_2 - u_3] = 0$$

$$150 \times 10^3 [-u_2 + u_3] = 3 \times 10^3$$

$$150 \times 10^3 \times u_2 = 3 \times 10^3$$

$$u_2 = \frac{3}{150}$$

$$\boxed{u_2 = 0.02 \text{ mm}}$$

$$150 \times 10^3 [0.04 - u_3] = 0$$

$$0.04 - u_3 = 0$$

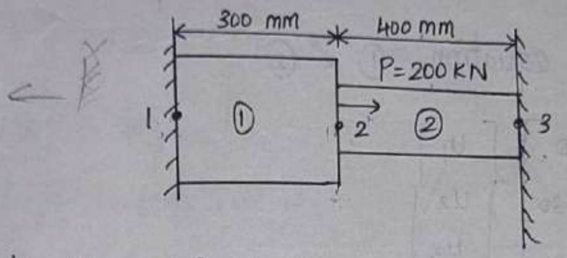
$$\boxed{u_3 = 0.04 \text{ mm}}$$

Elongation,

$$\delta_L = \frac{PL}{AE} = \frac{3\times10^3 \times 800}{300\times 2\times10^5}$$

$$\boxed{\delta_L = 0.04 \text{ mm}}$$

3. Consider a bar element as shown in fig, an axial load of 200 KN is applied at point P, Take $A_1 = 2400 \text{ mm}^2$, $E_1 = 70\times10^9 \text{ N/m}^2$, $A_2 = 600 \text{ mm}^2$, $E_2 = 200\times10^9 \text{ N/m}^2$



Calculate (i) Nodal displacement at P

(ii) Stresses at each element

(iii) Reaction force

Given

$A_1 = 2400 \text{ mm}^2$

$A_2 = 600 \text{ mm}^2$

$E_1 = 70\times10^9 \text{ N/m}^2 = 70\times10^3 \text{ N/mm}^2$

$E_2 = 200\times10^9 \text{ N/m}^2 = 200\times10^3 \text{ N/mm}^2$

$P = 200 \text{ KN} = 200\times10^3 \text{ N}$

Soln

1. For node 1 & 2 [Element 1]

Finite element equation

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{2400\times70\times10^3}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 10^4\begin{bmatrix} 56 & -56 \\ -56 & 56 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{—} ①$$

For node 2 & 3 [Element 2],

Finite element equation,

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{A_2 E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{600 \times 200 \times 10^3}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = 10^4 \begin{array}{c} \begin{array}{cc} a_{22} & a_{23} \end{array} \\ \begin{bmatrix} 30 & -30 \\ -30 & 30 \end{bmatrix} \\ \begin{array}{cc} a_{32} & a_{33} \end{array} \end{array} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \quad \text{--- (2)}$$

Assemble the finite element equation (1) & (2)

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 10^4 \begin{bmatrix} 56 & -56 & 0 \\ -56 & 86 & -30 \\ 0 & -30 & 30 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ F_2 \\ 0 \end{Bmatrix} = 10^4 \begin{bmatrix} 86 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix}$$

$$200 \times 10^3 = 10^4 \times (86u_2 - 30u_3)$$

$$100 \times 10^3 = 10^4 \times (-30u_2 + 30u_3)$$

Applying boundary conditions

$$F_2 = 200 \times 10^3$$

$$u_2 = ?$$

$$200 \times 10^3 = 86 \times 10^4 \, u_2$$

$$u_2 = 0.232 \text{ mm}$$

2. Stress in each element

$$\sigma_1 = E_1 \frac{du}{dx}$$

$$= 70 \times 10^3 \left(\frac{u_2 - u_1}{l} \right)$$

$$= 70 \times 10^3 \left(\frac{0.232 - 0}{300} \right)$$

$$\sigma_1 = 54.134 \text{ N/mm}^2$$

$$\sigma_2 = E_2 \frac{du}{dx}$$

$$= 200 \times 10^3 \left(\frac{u_3 - u_2}{l} \right) = 200 \times 10^3 \left(\frac{0 - 0.232}{400} \right)$$

$$\sigma_2 = -116 \text{ N/mm}^2$$

3. Reaction force

$$\{R\} = [K]\{u^*\} - \{F\}$$

$$= 1\times10^4 \begin{bmatrix} 56 & -56 & 0 \\ -56 & 86 & -30 \\ 0 & -30 & 30 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.232 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 200\times10^3 \\ 0 \end{Bmatrix}$$

$$= 10^4 \begin{bmatrix} -12.992 \\ 19.952 \\ -6.96 \end{bmatrix} - \begin{Bmatrix} 0 \\ 200\times10^3 \\ 0 \end{Bmatrix}$$

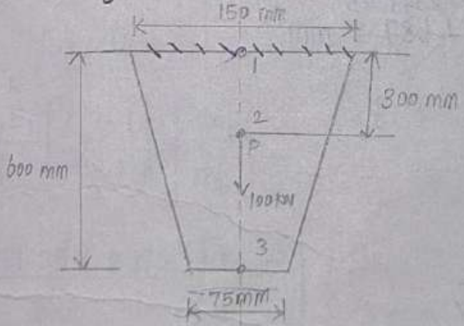
$$\{R_1\} = -12.992\times10^4 \text{ N}$$

$$\{R_2\} = 0.$$

$$\{R_3\} = -6.96\times10^4 \text{ N}$$

4. Consider a tapper steel plate of uniform thickness 25 mm as shown in fig. $E = 2\times10^5$ N/mm^2 and weight density $\rho = 0.82\times10^{-4}$ N/mm^3. In addition to its self weight, the plate is subjected to a point load of $P = 100$ N at its mid point. Calculate the following by modeling the plate with two finite elements.

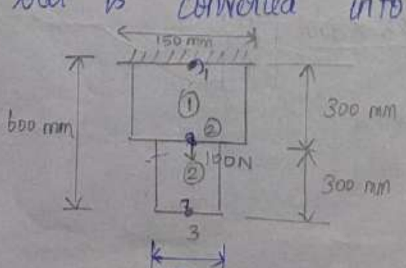
Data change



i) Global force vector (ii) Global Stiffness matrix (iii) Displacement at each element (iv) Stress at each element (v) Reaction force

Soln

Tapper bar is converted into stepped bar



Area at node 1 = Width × Thickness

$= 150 \times 25$

$= 3750 \text{ mm}^2$

Area at node 2 $= \left(\frac{W_1 + W_2}{2}\right) \times t$

$= \left(\frac{150+75}{2}\right) \times 25$

$= 2812.5 \text{ mm}^2$

Area at node 3 $= W_2 \times t$

$= 75 \times 25$

$= 1875 \text{ mm}^2$

Area of element ① $= \frac{\text{Node 1 Area} + \text{Node 2 Area}}{2}$

$= \frac{3750 + 2812.5}{2}$

$\bar{A}_1 = 3281.25 \text{ mm}^2$

Area of element ② $= \frac{\text{Node 2 Area} + \text{Node 3 Area}}{2}$

$= \frac{2812.5 + 1875}{2}$

$\bar{A}_2 = \cancel{4687.5}\ 2343.75 \text{ mm}^2$

$\rho = 0.82 \times 10^{-4} \text{ N/mm}^3$

$E = 2 \times 10^5 \text{ N/mm}^2$

$P = 100 \text{ N} = \cancel{100 \times 10^3 \text{ N}}$

Considering self weight,

Body force vector

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{\rho_1 A_1 l_1}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

For element ①,

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{0.82 \times 10^{-4} \times 3281.25 \times 300}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 40.359 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 40.359 \end{Bmatrix}$$

For element ②,

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{\rho_2 A_2 l_2}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= \frac{0.82 \times 10^{-4} \times 2343.75 \times 300}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= 28.828 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 28.828 \\ 28.828 \end{Bmatrix}$$

Global force vector,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 40.359 + 28.828 + 100 \\ 28.828 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

For element ①

Finite Element Equation,

$$\{F\} = [K]\{u^*\}$$ FE. EQU:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \overset{K}{\frac{A_1 E_1}{l_1}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{3281.25 \times 2 \times 10^5}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 2.1875 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 1 \times 10^6 \begin{bmatrix} 2.187 & -2.187 \\ -2.187 & 2.187 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{A_2 E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{2343.75 \times 2 \times 10^5}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = 1.5625 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 1 \times 10^6 \begin{bmatrix} 1.562 & -1.562 \\ -1.562 & 1.562 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Assemble the finite element equation

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 10^6 \begin{bmatrix} 2.187 & -2.187 & 0 \\ -2.187 & 3.749 & -1.562 \\ 0 & -1.562 & 1.562 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Applying the boundary conditions

$u_1 = 0 \qquad f_1 = 40.359$

$u_2 = ? \qquad f_2 = 169.187$

$u_3 = ? \qquad f_3 = 28.828$

$$10^6(3.749\,u_2 - 1.562\,u_3) = 169.187$$

$$10^6(-1.562\,u_2 + 1.562\,u_3) = 28.828$$

$$10^6 \times 2.187\,u_2 = 198.015$$

$$2.187\,u_2 = \frac{198.015}{10^6}$$

$$u_2 = \left(\frac{198.015}{10^6}\right) \div 2.187$$

$$u_2 = 9.05 \times 10^{-5}$$

$$u_3 = 1.08 \times 10^{-4}$$

Stress in each element,

$$\sigma_1 = E\frac{du}{dx}$$

$$= E \times \frac{(u_2 - u_1)}{l_1}$$

$$= \frac{2\times10^5\,(9.05\times10^{-5} - 0)}{300}$$

$$\sigma_1 = 0.06 \text{ N/mm}^2$$

$$\sigma_2 = E\frac{du}{dx}$$

$$= 2\times10^5\,\frac{(u_3 - u_2)}{l_2}$$

$$= 2\times10^5\,\frac{(1.07\times10^{-4} - 9.05\times10^{-5})}{300}$$

$$\sigma_2 = 0.011 \text{ N/mm}^2$$

Reaction force at its support

$$\{R\} = \{K\}\{u^*\} - \{F\}$$

$$= 1\times10^6 \begin{bmatrix} 2.187 & -2.187 & 0 \\ 2.187 & 3.75 & -1.562 \\ 0 & -1.562 & 1.562 \end{bmatrix} \begin{Bmatrix} 0 \\ 9.05\times10^{-5} \\ 1.08\times10^{-4} \end{Bmatrix} - \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

$$= 1\times10^5 \begin{bmatrix} -1.979\times10^{-3} \\ 1.706\times10^{-3} \\ 2.73\times10^{-4} \end{bmatrix} - \begin{Bmatrix} 40.359 \\ 169.187 \\ 28.828 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{bmatrix} -238.259 \\ 1.413 \\ -1.528 \end{bmatrix}$$

Trusses

A Truss is defined as a structure add up of several bars, joints are welded together.

The following assumptions are made while finding the forces in a Truss.

* All the members are pin jointed
* The Truss is loaded only at a Joint
* The self weight of the members are neglected unless stated.

The stiffness matrix,

$$[K] = \frac{A_e E_e}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

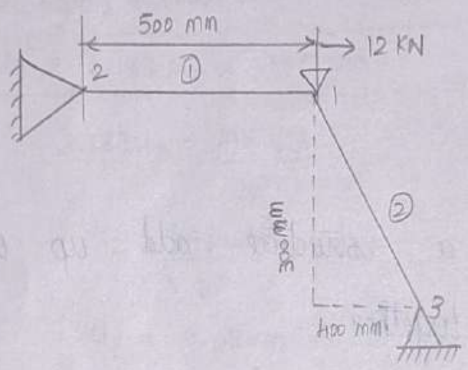
Stress,

$$[\sigma] = \frac{E}{l_e} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Applications

* Airport
* Railway track
* Bridges
* Cricket stadium

1. For the 2 bar Truss element as shown in fig, determine the displacements of node 1 and stress in element (node 1,3). Take $E = 70$ GPa, $A = 200\ mm^2$

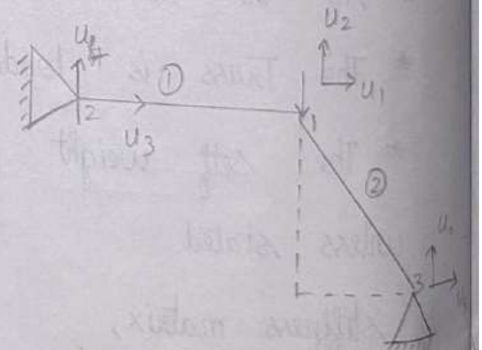


Soln

$E = 70 \times 10^9\ N/m^2$

$E = 70 \times 10^3\ N/mm^2$

For the coordinates,



(Node 1) = (0,0) → Take origin

(Node 2) = (-500,0)

(Node 3) = (400,-300)

For element ①,

$$le_1 = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$= \sqrt{(-500-0)^2 + (0-0)^2}$$

$$le_1 = 500\ mm$$

Direction cosines,

$$l_1 = \frac{x_2 - x_1}{le_1} = \frac{-500-0}{500} = -1$$

$$m_1 = \frac{y_2 - y_1}{le_1} = \frac{0-0}{500} = 0$$

For element ②,

$$le_2 = \sqrt{(x_3-x_1)^2 + (y_3-y_1)^2}$$

$$= \sqrt{(400-0)^2 + (-300-0)^2}$$

$$le_2 = 500 \text{ mm}$$

Direction cosines,

$$l_2 = \frac{x_3 - x_1}{le_2} = \frac{-400-0}{500} = 0.8$$

$$m_2 = \frac{y_3 - y_1}{le_2} = \frac{-300-0}{500} = -0.6$$

For element ① (Displacements 1, 2, 3, 4)

$$[K]_{e_1} = \frac{A_1 E_1}{le_1} \begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & l_1 m_1 \\ -l_1 m_1 & -m_1^2 & l_1 m_1 & m_1^2 \end{bmatrix}$$

$$= \frac{200 \times 70 \times 10^3}{500} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 28 \times 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

For element ② (Displacements u_1, u_2, u_5, u_6)

$$[K]_{e_2} = \frac{A_2 E_2}{le_2} \begin{bmatrix} l_2^2 & l_2 m_2 & -l_2^2 & -l_2 m_2 \\ l_2 m_2 & m_2^2 & -l_2 m_2 & -m_2^2 \\ -l_2^2 & -l_2 m_2 & l_2^2 & l_2 m_2 \\ -l_2 m_2 & -m_2^2 & l_2 m_2 & m_2^2 \end{bmatrix}$$

$$= \frac{200 \times 70 \times 10^3}{500} \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} & \\ \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & +0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array} \end{array}$$

$$[K]_{e_2} = 28 \times 10^3 \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

Assemble the stiffness matrix, $[K]$

$$K = 28 \times 10^3 \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 1+0.64 & 0-0.48 & -1 & 0 & -0.64 & 0.48 \\ 0-0.48 & 0+0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ 0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Finite element equation,

$$\{F\} = [K]\{u\}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

Applying boundary conditions,

$u_3 = u_4 = u_5 = u_6 = 0$

$F_2 = -12 \times 10^3$ N

$F_1 = 0$

$u_1 = ?$

$u_2 = ?$

$$\begin{Bmatrix} 0 \\ -12\times10^3 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 28\times10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ 0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Applying the boundary conditions in FEE,

$$\begin{Bmatrix} 0 \\ -12\times10^3 \end{Bmatrix} = 28\times10^3 \begin{bmatrix} 1.64 & -0.48 \\ -0.48 & 0.36 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Solving the above equations,

$$0 = 28\times10^3\left[1.64u_1 - 0.48u_2\right]$$

$$-12\times10^3 = 28\times10^3\left[-0.48u_1 + 0.36u_2\right]$$

$$u_1 = -0.57$$

$$u_2 = -1.95$$

Stress at element ①

$$\sigma_1 = \frac{E_1}{le_1}\begin{bmatrix} -l_1 & -m_1 & l_1 & m_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{70\times10^3}{500}\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} -0.57 \\ -1.95 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 140\left[-0.57+0+0+0\right]$$

$$\sigma_1 = -79.8 \ N/mm^2$$

$$\sigma_2 = \frac{E_2}{le_2}\begin{bmatrix} -l_2 & -m_2 & l_2 & m_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$= \frac{70\times10^3}{500}\begin{bmatrix} -0.8 & +0.6 & 0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} -0.57 \\ -1.95 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 140\left[0.456 - 1.17 + 0 + 0\right]$$

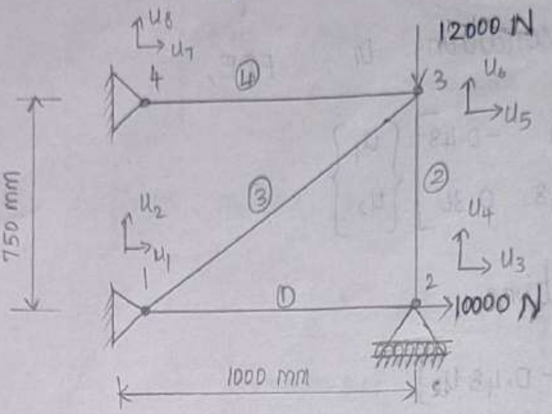
$$\sigma_2 = -99.96 \ N/mm^2$$

2. Consider a 4 bar truss as shown in fig. Take Youngs modulus $E = 2 \times 10^5$ N/mm^2 and $A = 625$ mm^2 for all elements. Determine

i) Element Stiffness matrix for each element

ii) Assemble the structural stiffness matrix [K] for the entire elements

iii) Solve the nodal displacements.



Soln

Consider Node 1 as Origin,

Node 1 : $(0, 0)$ — (x_1, y_1)

Node 2 : $(1000, 0)$ — (x_2, y_2)

Node 3 : $(1000, 750)$ — (x_3, y_3)

Node 4 : $(0, 750)$ — (x_4, y_4)

For element ①,

$$le_1 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$= \sqrt{(1000-0)^2+(0-0)^2}$$
$$le_1 = 1000 \text{ mm}$$

$$l_1 = \frac{x_2-x_1}{le_1} = \frac{1000-0}{1000} = 1$$
$$m_1 = \frac{y_2-y_1}{le_1} = \frac{0-0}{1000} = 0$$

For element ②

$$le_2 = \sqrt{(x_3-x_2)^2+(y_3-y_2)^2}$$
$$= \sqrt{(1000-1000)^2+(750-0)^2}$$
$$le_2 = 750 \text{ mm}$$

$$l_2 = \frac{x_3-x_2}{le_2} = \frac{1000-1000}{750} = 0$$
$$m_2 = \frac{y_3-y_2}{le_2} = \frac{750-0}{750} = 1$$

For element ③

$$le_3 = \sqrt{(x_3-x_1)^2+(y_3-y_1)^2}$$
$$= \sqrt{(1000-0)^2+(750-0)^2}$$
$$le_3 = 1250 \text{ mm}$$

$$l_3 = \frac{x_3-x_1}{le_3} = \frac{1000-0}{1250} = 0.8$$
$$m_3 = \frac{y_3-y_1}{le_3} = \frac{750-0}{1250} = 0.6$$

For element ④

$$l_{e_4} = \sqrt{(x_4-x_3)^2 + (y_4-y_3)^2}$$

$$= \sqrt{(0-1000)^2 + (750-750)^2}$$

$$l_{e_4} = 1000 \text{ mm}$$

$$l_4 = \frac{x_4 - x_3}{l_{e_4}} = \frac{0-1000}{1000} = -1$$

$$m_4 = \frac{y_4 - y_3}{l_{e_4}} = \frac{750-750}{1000} = 0$$

Stiffness matrix for element ①

$$[K]_{e_1} = \frac{A_1 E_1}{l_{e_1}} \begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & l_1 m_1 \\ -l_1 m_1 & -m_1^2 & l_1 m_1 & m_1^2 \end{bmatrix}$$

$$= \frac{625 \times 2 \times 10^5}{1000} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K]_{e_1} = 125 \times 10^3 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Stiffness matrix for element ②

$$[K]_{e_2} = \frac{A_2 E_2}{l_{e_2}} \begin{bmatrix} l_2^2 & l_2 m_2 & -l_2^2 & -l_2 m_2 \\ l_2 m_2 & m_2^2 & -l_2 m_2 & -m_2^2 \\ -l_2^2 & -l_2 m_2 & l_2^2 & l_2 m_2 \\ -l_2 m_2 & -m_2^2 & l_2 m_2 & m_2^2 \end{bmatrix}$$

$$[K]_{e_2} = 166.6 \times 10^3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Stiffness matrix for element ③

$$[K]_{e_3} = \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} 100 \times 10^3$$

Stiffness element matrix (4)

$$[K_e]_4 = 125 \times 10^3 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K_e]_1 = 10^3 \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ & \begin{bmatrix} 125 & 0 & -125 & 0 \\ 0 & 0 & 0 & 0 \\ -125 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

$$[K_e]_2 = 10^3 \begin{array}{c c} & \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 166.6 & 0 & -166.6 \\ 0 & 0 & 0 & 0 \\ 0 & -166.6 & 0 & 166.6 \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

$$[K_e]_3 = 10^3 \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ & \begin{bmatrix} 64 & 48 & -64 & -48 \\ 48 & 36 & 48 & -36 \\ -64 & -48 & 64 & 48 \\ -48 & -36 & 48 & 36 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array} \end{array}$$

$$[K_e]_4 = 10^3 \begin{array}{c c} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ & \begin{bmatrix} 125 & 0 & -125 & 0 \\ 0 & 0 & 0 & 0 \\ -125 & 0 & 125 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array}$$

Assemble the stiffness matrix [K]

$[K]_e = 10^3$

1	2	3	4	5	6	7	8	
189	48	-125	0	-64	-48	0	0	1
48	36	0	0	48	-36	0	0	2
-125	0	125	0	0	0	0	0	3
0	0	0	166.6	0	-166.6	0	0	4
-64	-48	0	0	189	48	-125	0	5
-48	-36	0	-166.6	48	202.6	0	0	6
0	0	0	0	-125	0	125	0	7
0	0	0	0	0	0	0	0	8

Finite element equation,

$$\{F\} = [K]\{u\}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}$$

Applying boundary conditions,

$$u_1 = u_2 = u_4 = u_7 = u_8 = 0$$

$$u_3 = ? \quad u_5 = ? \quad u_6 = ?$$

$$F_3 = 10000 \text{ N}$$

$$F_6 = -12000 \text{ N}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 10\times10^3 \\ 0 \\ 0 \\ -12\times10^3 \\ 0 \\ 0 \end{Bmatrix} = 10^3 \begin{bmatrix} 189 & 48 & -125 & 0 & -64 & -48 & 0 & 0 \\ 48 & 36 & 0 & 0 & 48 & -36 & 0 & 0 \\ -125 & 0 & 125 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 166.6 & 0 & -166.6 & 0 & 0 \\ -64 & -48 & 0 & 0 & 189 & 48 & -125 & 0 \\ -48 & -36 & 0 & -166.6 & 48 & 202.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -125 & 0 & 125 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ 0 \\ u_5 \\ u_6 \\ 0 \\ 0 \end{Bmatrix}$$

$$10\times10^3 = 10^3\left[125u_3\right]$$

$$0 = 10^3\left[189u_5 + 48u_6\right]$$

$$-12\times10^3 = 10^3\left[48u_5 + 202.6u_6\right]$$

$$u_3 = 0.08 \text{ mm}$$

$$u_5 = 0.013 \text{ mm}$$

$$u_6 = -0.06 \text{ mm}$$

Temperature effect

The nodal force vector,

$$\{F\} = EA\alpha\,\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Thermal stress,

$$\{\sigma\} = E \cdot \frac{du}{dx} - E\alpha\,\Delta T$$

$$\sigma_1 = E_1 \frac{(u_2 - u_1)}{l_1} - E_1 \alpha_1\,\Delta T$$

$$\sigma_2 = E_2 \frac{(u_3 - u_2)}{l_2} - E_2 \alpha_2\,\Delta T$$

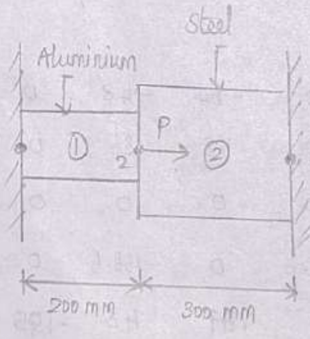
1. An Axial load of 4×10^5 N is applied at 30°c to the rod as shown in the fig. The temperature is raised to 60°c. Calculate the following. (i) Assemble the stiffness matrix [K] and force vector, F
   (ii) Nodal displacements
   (iii) Stresses in each element
   (iv) Reaction force at each nodal point

For Aluminium,

$A_1 = 1000\ mm^2$

$E_1 = 0.7 \times 10^5\ N/mm^2$

$\alpha_1 = 23 \times 10^{-6} /°C$



For Steel,

$A_2 = 1500\ mm^2$

$E_2 = 2 \times 10^5\ N/mm^2$

$\alpha_2 = 12 \times 10^{-6} /°C$

Solution

For the Finite Element equation at Element ①

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{A_1 E_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1000 \times 0.7 \times 10^5}{200} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} 3.5 & -3.5 \\ -3.5 & 3.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{— ①}$$

For the Finite Element Equation at Element ②

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \frac{A_2 E_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1500 \times 2 \times 10^5}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \quad \text{— ②}$$

Assemble the Finite Element Equation,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} 3.5 & -3.5 & 0 \\ -3.5 & 13.5 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Assembling force vector,

$$\{F\} = EA\alpha\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

For element ①

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = E_1 \times A_1 \times \alpha_1 \times \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= 0.7 \times 10^5 \times 1000 \times 23 \times 10^{-6} \times 30 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = 1 \times 10^5 \begin{bmatrix} -0.483 \\ 0.483 \end{bmatrix}$$

For element ②

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = E_2 \times A_2 \times \alpha_2 \times \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= 2 \times 10^5 \times 1500 \times 12 \times 10^{-6} \times 30 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = 1 \times 10^5 \begin{Bmatrix} -1.08 \\ 1.08 \end{Bmatrix}$$

Assemble the Force Vector,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 1 \times 10^5 \begin{Bmatrix} -0.483 \\ 0.483 - 1.08 + 4 \\ 1.08 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 1\times10^5 \begin{Bmatrix} -0.483 \\ 3.403 \\ 1.08 \end{Bmatrix}$$

Applying boundary conditions,

$u_1 = u_3 = 0, \quad u_2 = ?$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = 1\times10^5 \begin{bmatrix} 3.5 & -3.5 & 0 \\ -3.5 & 13.5 & 0 \\ 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix}$$

$$1\times10^5 \begin{Bmatrix} -0.483 \\ 3.403 \\ 1.08 \end{Bmatrix} = 1\times10^5 \begin{bmatrix} 3.5 & -3.5 & 0 \\ -3.5 & 13.5 & 0 \\ 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix}$$

$$1\times10^5 \times (3.403) = 1\times10^5 \times 13.5\,(u_2)$$

$$u_2 = \frac{3.403}{13.5}$$

$$u_2 = 0.252 \text{ mm}$$

Thermal Stresses,

Stress at element ①

$$\sigma_1 = \frac{E_1(u_2-u_1)}{l_1} - E_1\alpha_1\Delta T$$

$$= 0.7\times10^5 \frac{(0.252-0)}{200} - (0.7\times10^5 \times 23\times10^{-6} \times 30)$$

$$\sigma_1 = 39.2 \text{ N/mm}^2 \text{ (Tensile stress)}$$

Stress at element ②

$$\sigma_2 = \frac{E_2(u_3-u_2)}{l_2} - E_2\alpha_2\Delta T$$

$$= 2\times10^5 \frac{(0-0.252)}{300} - (2\times10^5 \times 12\times10^{-6} \times 30)$$

$$\sigma_2 = -238.66 \text{ N/mm}^2 \text{ (compressive stress)}$$

Reaction forces,

$$\{R\} = \{K\}\{u^*\} - \{F\}$$

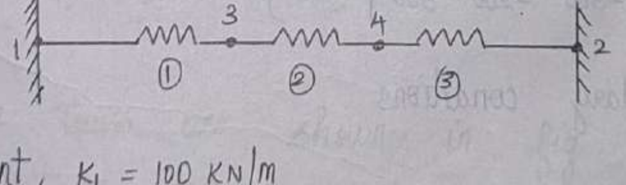
$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 1\times10^5 \begin{bmatrix} 3.5 & -3.5 & 0 \\ -3.5 & 13.5 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.252 \\ 0 \end{Bmatrix} - 1\times10^5 \begin{Bmatrix} -0.483 \\ 3.403 \\ 1.08 \end{Bmatrix}$$

$$= 1\times10^5 \begin{bmatrix} -0.882 \\ 3.402 \\ -2.52 \end{bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = 1\times10^5 \begin{bmatrix} -0.3993 \\ 0 \\ -3.601 \end{bmatrix}$$

Trusses

3. A spring assemblage with an arbitrary numbering nodes as shown in fig. The nodes ① and ② are fixed and a force of 500 KN is applied at the node 4 in the direction of 'x'. Calculate (i) Global stiffness Matrix [K]
(ii) Nodal displacement (u_1, u_2, u_3, u_4)
(iii) Reaction force at each nodal point



Spring constant, $K_1 = 100$ KN/m
$K_2 = 200$ KN/m
$K_3 = 300$ KN/m

Solution

Finite Element Equation for element ① (Node 1 & 3).

$$\begin{Bmatrix} F_1 \\ F_3 \end{Bmatrix} = 100 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix}$$

$$= \begin{array}{cc} 1 & 3 \end{array} \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{array}{c} 1 \\ 3 \end{array} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix}$$

For element ②, (Node 3 & 4)

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = 200 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{cc} 3 & 4 \end{array} \\ & \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{array}{c} 3 \\ 4 \end{array} \end{array} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

For element ③, (Node 4 & 2)

$$\begin{Bmatrix} F_4 \\ F_2 \end{Bmatrix} = 300 \begin{array}{c} \begin{array}{cc} 4 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} 4 \\ 2 \end{array} \end{array} \begin{Bmatrix} u_4 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_2 \end{Bmatrix}$$

Assemble the stiffness matrix equation,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Applying boundary conditions

$u_1 = u_2 = 0$

$u_3 = ?\quad u_4 = ?$

$F_1 = F_2 = F_3 = 0$

$F_4 = 500 \times 10^3$ N

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 500 \times 10^3 \end{Bmatrix} = \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$0 = 300\,u_3 - 200\,u_4$$

$$\underline{500\times10^3 = -200\,u_3 + 500\,u_4}$$

Calc $u_3 = 909.09 \text{ mm} = 0.909 \text{ m}$

$u_4 = 1363.63 \text{ mm} = 1.363 \text{ m}$

Reaction forces,

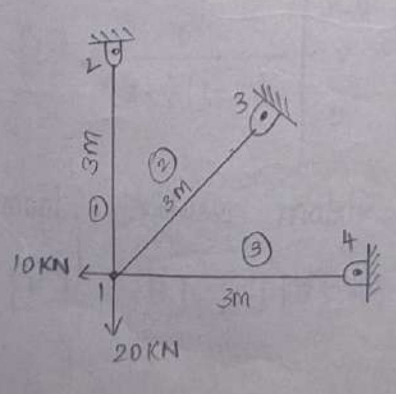
$$\{R\} = [K]\{u^*\} - \{F\}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} 100 & 0 & -100 & 0 \\ 0 & 300 & 0 & -300 \\ -100 & 0 & 300 & -200 \\ 0 & -300 & -200 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.909 \\ 1.363 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 500\,\cancel{\times10^3} \end{Bmatrix}$$

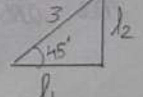
$$= \begin{bmatrix} -90.9 \\ -408.9 \\ 0.1 \\ 499.7 \end{bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 500\,\cancel{\times10^3} \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} -90.9 \\ -408.9 \\ 0.1 \\ 0 \end{bmatrix}$$

4. For the plane Truss as shown in fig, the Youngs modulus of the element, $E = 201$ GPa and Area of the element, $A = 4\times10^{-4}\ m^2$. Determine the Horizontal & Vertical displacements of all the nodal points and stresses in each element.



Consider Node 1 as Origin



Node 1 : $(0,0)$

Node 2 : $(0,3)$

Node 3 : $(2.12, 2.12)$

Node 4 : $(3,0)$

$l_1 = 2.12$

$\cos 45 = \frac{l}{3}$

$l_1 = \cos 45 \times 3$

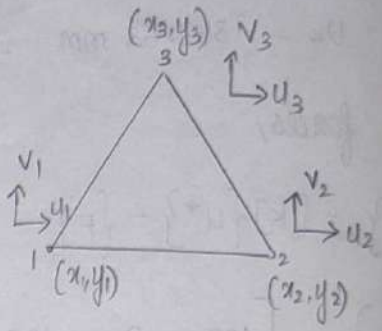
$= 2.12$ m

UNIT-3

TWO DIMENSIONAL PROBLEMS

CST

Constant Strain Triangular Elements
Under,



Plane stress condition
Plane strain condition

Calculate,

1. Stress-Strain relationship matrix [D]
2. Stiffness matrix [k]
3. Stresses $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$
4. Shape function
5. Temperature effects

Plane stress

Plane stress is defined to be a state of stress in which normal stress [σ] and shear stress [τ], directed to the plane assumed to be Zero.

$$\sigma_z = 0$$

$$\tau_{xz} = \tau_{yz} = 0$$

Plane strain

Plane strain is defined to be state of strain in which the strain normal to the xy plane and shear strain are assumed to be zero.

$$e_z = 0$$

$$\gamma_{xz} = \gamma_{yz} = 0$$

Formulae used

1. For Constant Strain Triangular (CST) element

Shape function, $N_1 + N_2 + N_3 = 1$

Co-Ordinate, $x = N_1 x_1 + N_2 x_2 + N_3 x_3$

Co-Ordinate, $y = N_1 y_1 + N_2 y_2 + N_3 y_3$

(or)

Co-Ordinate, $x = (x_1 - x_3) N_1 + (x_2 - x_3) N_2 + x_3$

Co-Ordinate, $y = (y_1 - y_3) N_1 + (y_2 - y_3) N_2 + y_3$

2. Area of the Triangular element

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

3. Strain Displacement matrix for CST element

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

where,

$q_1 = y_2 - y_3$; $q_2 = y_3 - y_1$; $q_3 = y_1 - y_2$

$r_1 = x_3 - x_2$; $r_2 = x_1 - x_3$; $r_3 = x_2 - x_1$

4. Stress strain relationship matrix for Plane stress Problem

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

ν - Poison ratio

5. Stress-strain relationship matrix for Plane strain Problem

$$[D] = \frac{1}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

6. Element Stiffness matrix for CST element

$$[K] = [B]^T [D] [B] A t$$

7. Element stress

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D][B] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$\sigma_x, \sigma_y \rightarrow$ Normal stresses

$\tau_{xy} \rightarrow$ Shear stress

$u, v \rightarrow$ Nodal displacements

$$\{\sigma\} = [D][B]\{u\}$$

8. Maximum normal stress

$$\sigma_{max} = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Minimum normal stress

$$\sigma_{min} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

9. Principle angle

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

10. Element strain

$$\{e\} = [B]\{u\}$$

$$= [B] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

11. Temperature effects,

Initial strain $\{e_0\}$
Plane stress problem $= \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix}$

Initial strain $\{e_0\}$
Plane strain problem $= (1+\nu) \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix}$

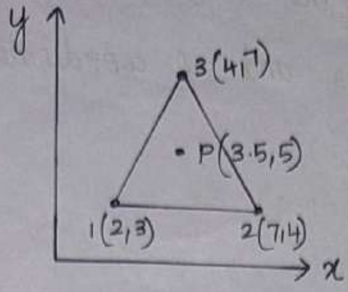
$\alpha \rightarrow$ Co-efficient of Thermal expansion

$\nu \rightarrow$ Poisson's ratio

12. Element Temperature Force

$$\{f\} = [B]^T [D] \{e_0\} t A$$

1. Determine the shape function N_1, N_2, N_3 at the interior point P for the triangular element



<u>Solution</u>

Co-ordinates,

$x_1 = 2, \; y_1 = 3$

$x_2 = 7, \; y_2 = 4$

$x_3 = 4, \; y_3 = 7$

$x = 3.5, \; y = 5$

Shape function for CST elements

$N_1 + N_2 + N_3 = 1$ — ①

$x = (x_1 - x_3) N_1 + (x_2 - x_3) N_2 + x_3$

$y = (y_1 - y_3) N_1 + (y_2 - y_3) N_2 + y_3$

Sub the coordinate values in the above eqns,

$x = (2-4) N_1 + (7-4) N_2 + 4$

$y = (3-7) N_1 + (4-7) N_2 + 7$

$x = -2N_1 + 3N_2 + 4$

$y = -4N_1 - 3N_2 + 7$

$N_1 = 0.416$

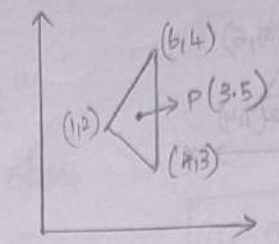
$N_2 = 0.111$

① $\Rightarrow 0.416 + 0.111 + N_3 = 1$

$N_3 = 1 - 0.416 - 0.111$

$N_3 = 0.4723$

2. A Nodal coordinates of the Triangular element are shown in fig at the interior point 'P'. The x-coordinate is 3.5 and $N_1 = 0.4$ Calculate shape functions N_2 & N_3 and Y coordinates at the point P



Given

$x_1 = 1,\ y_1 = 2$

$x_2 = 4,\ y_2 = 3$

$x_3 = 6,\ y_3 = 4$

$x = 3.5,\ y = P$

Soln

W.K.T,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$3.5 = N_1(1) + N_2(4) + N_3(6)$$

$$3.5 = N_1 + 4N_2 + 6N_3$$

$$3.5 = 0.4 + 4N_2 + 6N_3$$

$$4N_2 + 6N_3 = 3.1 \quad \text{—— (1)}$$

$$N_1 + N_2 + N_3 = 1$$

$$N_2 + N_3 = 1 - 0.4$$

$$N_2 + N_3 = 0.6 \quad \text{—— (2)}$$

Solving (1) & (2)

$$N_2 = 0.25$$

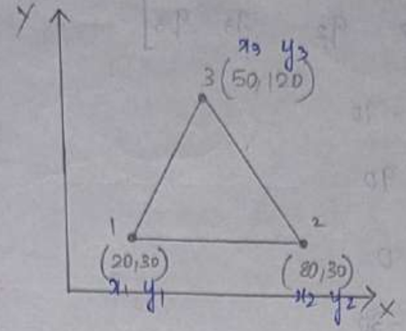
$$N_3 = 0.35$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$y = 0.4(2) + 0.25(3) + 0.35(4)$$

$$y = 2.95$$

3. Determine the Stiffness matrix for the CST element. The coordinates are given in units of mm. Assume Plane stress condition. Take $E = 210$ GPa, Poisson ratio, $\nu = 0.25$ and thickness $t = 10$ mm



Given

$E = 210$ GPa $= 210 \times 10^3$ N/mm^2

$\nu = 0.25$

$t = 10$ mm

To find

Stiffness Matrix [K]

Solution

Stiffness matrix,

$[K] = [B]^T [D] [B] \times A \times t$

$[B] \rightarrow$ Strain displacement matrix

$[D] \rightarrow$ Stress-strain relationship matrix

$A \rightarrow$ Area

$t \rightarrow$ Thickness

$$\text{Area}(A) = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 20 & 30 \\ 1 & 80 & 30 \\ 1 & 50 & 120 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(9600-1500) - 20(120-30) + 30(50-80)\right]$$

$A = 2700$ mm^2

Strain displacement matrix,

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$q_1 = y_2 - y_3 = 30 - 120 = -90$

$q_2 = y_3 - y_1 = 120 - 30 = 90$

$q_3 = y_1 - y_2 = 30 - 30 = 0$

$r_1 = x_3 - x_2 = 50 - 80 = -30$

$r_2 = x_1 - x_3 = 20 - 50 = -30$

$r_3 = x_2 - x_1 = 80 - 20 = 60$

$$= \frac{1}{2 \times 2700}\begin{bmatrix} -90 & 0 & 90 & 0 & 0 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -90 & 30 & 90 & 60 & 0 \end{bmatrix}$$

$$= \frac{1}{5400} \times 30 \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

$$[B] = 5.55 \times 10^{-3}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -3 & -1 & 3 & 2 & 0 \end{bmatrix}$$

Stress strain relationship matrix,

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{210 \times 10^3}{1 - 0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= 224 \times 10^3 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= 224\times10^{3}\times 0.25\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 0\\ 0 & 0 & 1.5\end{bmatrix}$$

$$[D] = 56\times10^{3}\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 0\\ 0 & 0 & 1.5\end{bmatrix}$$

$$[D][B] = 56\times10^{3}\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 0\\ 0 & 0 & 1.5\end{bmatrix}\times 5.55\times10^{-3}\begin{bmatrix}-3 & 0 & 3 & 0 & 0 & 0\\ 0 & -1 & 0 & -1 & 0 & 2\\ -1 & -3 & -1 & 3 & 2 & 0\end{bmatrix}$$

$$= 310.8\begin{bmatrix}-12+0+0 & 0-1+0 & 12+0+0 & 0-1+0 & 0+0+0 & 0+2+0\\ -3+0+0 & 0-4+0 & 3+0+0 & 0-4+0 & 0+0+0 & 0+8+0\\ 0+0-1.5 & 0+0-4.5 & 0+0-1.5 & 0+0+4.5 & 0+0+3 & 0+0+0\end{bmatrix}$$

$$[D][B] = 310.8\begin{bmatrix}-12 & -1 & 12 & -1 & 0 & 2\\ -3 & -4 & 3 & -4 & 0 & 8\\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0\end{bmatrix}$$

$$[B]^{T}[D][B] = 5.55\times10^{-3}\begin{bmatrix}-3 & 0 & -1\\ 0 & -1 & -3\\ 3 & 0 & -1\\ 0 & -1 & 3\\ 0 & 0 & 2\\ 0 & 2 & 0\end{bmatrix}\times 310.8\begin{bmatrix}-12 & -1 & 12 & -1 & 0 & 2\\ -3 & -4 & 3 & -4 & 0 & 8\\ -1.5 & -4.5 & -1.5 & 4.5 & 3 & 0\end{bmatrix}$$

$$= 1.724\begin{bmatrix}37.5 & 7.5 & -34.5 & -1.5 & -3 & -6\\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8\\ -34.5 & 1.5 & 37.5 & -7.5 & -3 & -6\\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8\\ -3 & -9 & -3 & 9 & 6 & 0\\ -6 & -8 & -6 & -8 & 0 & 16\end{bmatrix}$$

$$[K] = [B]^{T}[D][B]\times A\times t$$

$$[K] = 46.15\times10^{3}\begin{bmatrix} 37.5 & 7.5 & -34.5 & -1.5 & -3 & -6 \\ 7.5 & 17.5 & 1.5 & -9.5 & -9 & -8 \\ -34.5 & 1.5 & 37.5 & -7.5 & -3 & 6 \\ -1.5 & -9.5 & -7.5 & 17.5 & 9 & -8 \\ -3 & -9 & -3 & 9 & 6 & 0 \\ -6 & -8 & 6 & -8 & 0 & 16 \end{bmatrix}$$

4. Calculate the element stresses σ_x, σ_y, τ_{xy}, σ_1 and σ_2 and the principle angle θ_P for the element as shown in fig.

The nodal displacements are

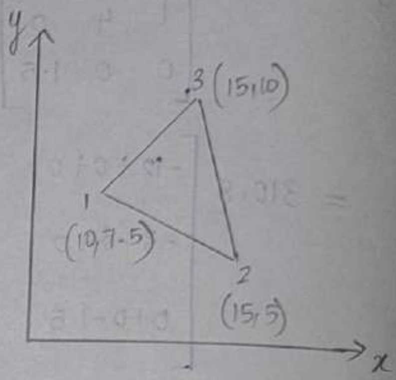
$u_1 = 2mm$, $v_1 = 1mm$

$u_2 = 0.5mm$, $v_2 = 0mm$

$u_3 = 3mm$, $v_3 = 1mm$

Take, $E = 2.1\times10^5\ N/mm^2$

$\nu = 0.25$.



Assume plane stress

<u>Given</u>

$x_1 = 10$, $y_1 = 7.5$

$x_2 = 15$, $y_2 = 5$

$x_3 = 15$, $y_3 = 10$

$u_1 = 2mm$, $v_1 = 1mm$

$u_2 = 0.5mm$, $v_2 = 0mm$

$u_3 = 3mm$, $v_3 = 1mm$

$E = 2.1\times10^5\ N/mm^2$

$\nu = 0.25$

<u>To find</u>

σ_x, σ_y, τ_{xy}, σ_1, σ_2 and θ_P

<u>Solution</u>

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 10 & 7.5 \\ 1 & 15 & 5 \\ 1 & 15 & 10 \end{vmatrix}$$

$$= \frac{1}{2}\left[75-50\right]$$

$$A = 12.5 \ mm^2$$

Element stresses,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D][B]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$q_1 = y_2 - y_3 = -5$ | $r_1 = x_3 - x_2 = 0$

$q_2 = y_3 - y_1 = 2.5$ | $r_2 = x_1 - x_3 = -5$

$q_3 = y_1 - y_2 = 2.5$ | $r_3 = x_2 - x_1 = 5$

$$[B] = \frac{1}{25}\begin{bmatrix} -5 & 0 & 2.5 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \\ 0 & -5 & -5 & 2.5 & 5 & 2.5 \end{bmatrix}$$

$$= \frac{2.5}{25}\begin{bmatrix} -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 1 & 2 & 1 \end{bmatrix}$$

$$[B] = 0.1\begin{bmatrix} -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -2 & -2 & 1 & 2 & 1 \end{bmatrix}$$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{2.1\times10^5}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$[D] = 56\times10^3\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 0\\ 0 & 0 & 1.5\end{bmatrix}$$

$$[D][B] = 56\times10^3\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 0\\ 0 & 0 & 1.5\end{bmatrix}\times 0.1\begin{bmatrix}-2 & 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & -2 & 0 & 2\\ 0 & -2 & -2 & 1 & 2 & 1\end{bmatrix}$$

$$= 5.6\times10^3\begin{bmatrix}-8 & 0 & 4 & -2 & 4 & 2\\ -2 & 0 & 1 & -8 & 1 & 8\\ 0 & -3 & -3 & 1.5 & 3 & 1.5\end{bmatrix}$$

Stresses,

$$\sigma = [D][B]\{u\}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = 5.6\times10^3\begin{bmatrix}-8 & 0 & 4 & -2 & 4 & 2\\ -2 & 0 & 1 & -8 & 1 & 8\\ 0 & -3 & -3 & 1.5 & 3 & 1.5\end{bmatrix}\begin{Bmatrix}2\\ 1\\ 0.5\\ 0\\ 3\\ 1\end{Bmatrix}$$

$$= 5.6\times10^3\begin{bmatrix}0\\ 7.5\\ 6\end{bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix}0\\ 42000\\ 33600\end{bmatrix}$$

Max stress,

$$\sigma_1 = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{0+42000}{2} + \sqrt{\left(\frac{0-42000}{2}\right)^2 + (33600)^2}$$

$$\sigma_1 = 60.67\times10^3\ N/mm^2$$

Min stress,

$$\sigma_2 = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{0+42000}{2} - \sqrt{\left(\frac{0-42000}{2}\right)^2 + (33600)^2}$$

$$\sigma_2 = -18.62 \times 10^3 \ N/mm^2$$

Principle angle,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$= \frac{2 \times 33600}{0 - 42000}$$

$$\tan 2\theta_p = -1.6$$

$$2\theta_p = \tan^{-1}(-1.6)$$

$$\theta_p = \frac{\tan^{-1}(-1.6)}{2}$$

$$\theta_p = -28.99$$

5. Nodal displacements are,

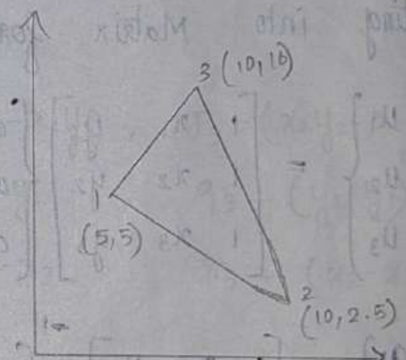
$u_1 = 1.5$ mm , $v_1 = 1$ mm

$u_2 = 0$ mm , $v_2 = 0.5$ mm

$u_3 = 1.5$ mm , $v_3 = 2$ mm

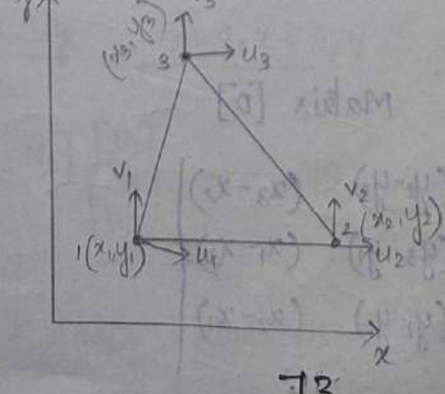
Take, $E = 2.1 \times 10^5$

$\nu = 0.25$



<u>Derivation of Shape function for the CST element</u>

Consider a typical CST element (constant strain Triangle) with nodes 1, 2, 3 and the Nodal displacement u_1, u_2, u_3 and v_1, v_2, v_3.



$$\text{Displacement } \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} \begin{matrix} \} \rightarrow x \text{ direction} \\ \\ \} \rightarrow y \text{ direction} \end{matrix}$$

CST element has got two D.O.F at each node (u, v). So the Total D.O.F is 6

Hence the generalised coordinates are,

$$u = a_1 + a_2 x + a_3 y$$

$$v = a_4 + a_5 x + a_6 y$$

Now, apply the Nodal displacement coordinates

$$u_1 = a_1 + a_2 x_1 + a_3 y_1 \qquad v_1 = a_4 + a_5 x_1 + a_6 y_1$$

$$u_2 = a_1 + a_2 x_2 + a_3 y_2 \qquad v_2 = a_4 + a_5 x_2 + a_6 y_2$$

$$u_3 = a_1 + a_2 x_3 + a_3 y_3 \qquad v_3 = a_4 + a_5 x_3 + a_6 y_3$$

Converting into Matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Take,

$$[D] = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$[D]^{-1} = \frac{C^T}{|D|}$$

$C \rightarrow$ Co-factors of Matrix $[D]$

$$C = \begin{vmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{vmatrix}$$

$$|D| = (x_2y_3 - x_3y_2) - (x_1y_3 - x_1y_2) + (y_1x_3 - y_1x_2)$$

Area of the CST element,

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$2A = (x_2y_3 - x_3y_2) - (x_1y_3 - x_1y_2) + (y_1x_3 - y_1x_2)$$

$$[D] = \frac{1}{2A} \times C^T$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_2y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_2 - y_1) \\ (x_3 - x_2) & (x_2 - x_3) & (x_2 - x_1) \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} P_1 & P_2 & P_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

where, $P_1 = (x_2y_3 - x_3y_2)$, $P_2 = (x_3y_1 - x_1y_3)$, $P_3 = (x_2y_2 - x_2y_1)$

$q_1 = y_2 - y_3$, $q_2 = y_3 - y_1$, $q_3 = (y_2 - y_1)$

$r_1 = x_3 - x_2$, $r_2 = x_2 - x_3$, $r_3 = (x_2 - x_1)$

W.K.T,

$$u = a_1 + a_2x + a_3y$$

$$u = \begin{bmatrix} 1 & x & y \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & x & y \end{bmatrix}\frac{1}{2A}\begin{bmatrix} P_1 & P_2 & P_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \frac{P_1 + q_1x + r_1y}{2A} \quad \frac{P_2 + q_2x + r_2y}{2A} \quad \frac{P_3 + q_3x + r_3y}{2A}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{--- (1)}$$

where,

$N_1, N_2, N_3 \rightarrow$ Shape functions

Similarly,

$$v = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \quad \text{--- } ②$$

Arrange the equations ① & ②

$$u = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

where,

Shape functions, $N_1 = \dfrac{p_1 + q_1 x + r_1 y}{2A}$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = \frac{p_3 + q_3 x + r_3 y}{2A}$$

Strain Displacement Matrix [B]

W.K.T

Strain in x direction,

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} \cdot u_2 + \frac{\partial N_3}{\partial x} u_3$$

Strain in y direction,

$$\varepsilon_y = \frac{\partial u}{\partial y} = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3$$

Strain along x-y direction

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$= \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} \cdot u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} \cdot v_3$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

W.K.T,

$$N_1 = \frac{P_1 + q_1 x_1 + r_1 y}{2A} \qquad N_2 = \frac{P_2 + q_2 x + r_2 y}{2A} \qquad N_3 = \frac{P_3 + q_3 x + r_3 y}{2A}$$

$$\frac{\partial N_1}{\partial x} = \frac{q_1}{2A} \qquad \frac{\partial N_2}{\partial x} = \frac{q_2}{2A} \qquad \frac{\partial N_3}{\partial x} = \frac{q_3}{2A}$$

$$\frac{\partial N_1}{\partial y} = \frac{r_1}{2A} \qquad \frac{\partial N_2}{\partial y} = \frac{r_2}{2A} \qquad \frac{\partial N_3}{\partial y} = \frac{r_3}{2A}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

This is the form of

$$\{\varepsilon\} = [B]\{u\}$$

Stress displacement matrix,

$$\{\sigma\} = [D]\{\varepsilon\}$$

$$= [D][B]\{u\}$$

Stiffness matrix,

$$[K] = \int_v [B]^T [D] \cdot [B] \cdot dv$$

$$= [B]^T [D] [B] A \cdot t$$

$t \rightarrow$ Thickness of the body

Axisymmetric elements

* The special two dimensional elements called Axisymmetric elements.

* Many 3-D problems in Engineering field symmetry about the axis of rotation. Such types of problems are called Axisymmetric problems.

* These problems can be solved by using 2-D elements. These axisymmetric problems are most conveniently described in Cylindrical coordinates (r, θ, z)

The required conditions for the problems to be axisymmetric

⇒ The problem domain must be symmetric about the axis of revolution, which is conventionally taken as the z-axis.

⇒ All the boundary conditions must be symmetric about the axis of revolution

⇒ All the loading conditions must be symmetric about the axis of revolution.

$$\text{Stress } \{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} \begin{matrix} \rightarrow \text{Radial stress} \\ \rightarrow \text{Longitudinal stress} \\ \rightarrow \text{Circumferential stress} \\ \rightarrow \text{Shear stress} \end{matrix}$$

$$\text{Strain } \{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}, \quad \text{Force } \{F\} = \begin{Bmatrix} F_r \\ F_z \end{Bmatrix}$$

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

Strain Displacement Matrix,

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$r = \frac{r_1 + r_2 + r_3}{3}, \quad z = \frac{z_1 + z_2 + z_3}{3}$$

$\alpha_1 = r_2 z_3 - r_3 z_2$	$\beta_1 = z_2 - z_3$	$\gamma_1 = r_3 - r_2$
$\alpha_2 = r_3 z_1 - r_1 z_3$	$\beta_2 = z_3 - z_1$	$\gamma_2 = r_1 - r_3$
$\alpha_3 = r_1 z_2 - r_2 z_1$	$\beta_3 = z_1 - z_2$	$\gamma_3 = r_2 - r_1$

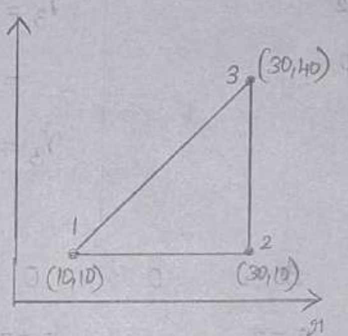
Stress-strain relationship matrix,

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Stiffness matrix,

$$[K] = 2\pi r A [B]^T [D] [B]$$

1. The nodal coordinates for an Axisymmetric triangular elements are given by $r_1 = 10$ mm, $z_1 = 10$ mm, $r_2 = 30$ mm, $z_2 = 10$ mm, $r_3 = 30$ mm and $z_3 = 40$ mm. Calculate the strain-Displacement Matrix and stiffness matrix. Take $E = 210$ GPa and $\nu = 0.25$



Given

$r_1 = 10$ mm, $z_1 = 10$ mm

$r_2 = 30$ mm, $z_2 = 10$ mm

$r_3 = 30$ mm, $z_3 = 40$ mm

To find

1. Strain displacement Matrix [B]
2. Stiffness Matrix [K]

Solution

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 1 & 10 & 10 \\ 1 & 30 & 10 \\ 1 & 30 & 40 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(1200-300) - 10(40-10) - 10(30-30)\right]$$

Area $= 300$ mm^2

$$r = \frac{r_1+r_2+r_3}{3} = \frac{10+30+30}{3}$$

$$r = 23.3 \text{ mm}$$

$$Z = \frac{Z_1+Z_2+Z_3}{3} = \frac{10+10+40}{3}$$

$$Z = 20 \text{ mm}$$

$$\alpha_1 = r_2 Z_3 - r_3 Z_2 = 1200 - 300$$

$$\alpha_1 = 900$$

$$\alpha_2 = r_3 Z_1 - r_1 Z_3 = 300 - 400$$

$$\alpha_2 = -100$$

$$\alpha_3 = r_1 Z_2 - r_2 Z_1 = 100 - 300$$

$$\alpha_3 = -200$$

$$\beta_1 = Z_2 - Z_3 = 10 - 40$$

$$\beta_1 = -30$$

$$\beta_2 = Z_3 - Z_1 = 40 - 10$$

$$\beta_2 = 30$$

$$\beta_3 = Z_1 - Z_2 = 10 - 10$$

$$\beta_3 = 0$$

$$\gamma_1 = r_3 - r_2 = 30 - 30$$

$$\gamma_1 = 0$$

$$\gamma_2 = r_1 - r_3 = 10 - 30$$

$$\gamma_2 = -20$$

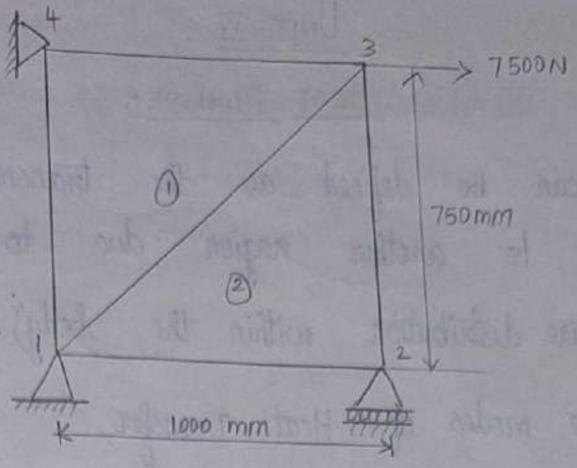
$$\gamma_3 = r_2 - r_1 = 30 - 10$$

$$\gamma_3 = 20$$

$$[B] = \frac{1}{600}\begin{bmatrix} -30 & 0 & 30 & 0 & 0 & 0 \\ 8.57 & 0 & 8.57 & 0 & 8.57 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -30 & -20 & 30 & 20 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0.014 & 0 & 0.014 & 0 & 0.014 & 0 \\ 0 & 0 & 0 & -0.03 & 0 & 0.03 \\ 0 & -0.05 & -0.03 & 0.05 & 0.03 & 0 \end{bmatrix}$$

2. The two Dimensional Beam as shown in fig can be divided into two CST elements. Determine the Nodal displacement and element stresses using plane stress condition. The body force is neglected in comparison with the external force. Take element thickness $t = 10$ mm, $E = 2\times10^5$ N/mm^2, Poisson ratio $= 0.25$



Given

$t = 10\ mm$

$E = 2\times10^5\ N/mm^2$

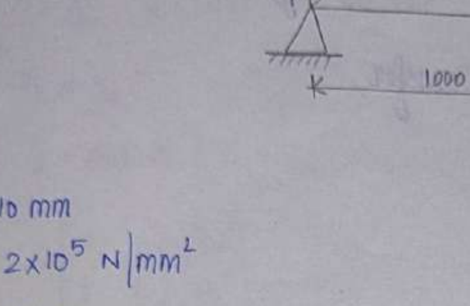
$\nu = 0.25$

To find

1. Nodal displacements $(u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4)$
2. Element stresses $(\sigma_1\ \&\ \sigma_2)$

Solution

Element ①



Take Node 1 as Origin

Node 1 $= (\overset{x_1}{0}, \overset{y_1}{0})$

Node 3 $= (\overset{x_2}{1000}, \overset{y_2}{750})$

Node 4 $= (\overset{x_3}{0}, \overset{y_3}{750})$

$$\text{Area, } A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1000 & 750 \\ 1 & 0 & 750 \end{vmatrix}$$

$$= \frac{1}{2}\left[(750000)\right]$$

$$A = 375000\ mm^2$$

UNIT-IV

HEAT TRANSFER

* Heat transfer can be defined as the transmission of energy from one region to another region due to temperature difference (Temperature distribution within the body).

* There are three modes of Heat Transfer,

⇒ Conduction

⇒ Convection

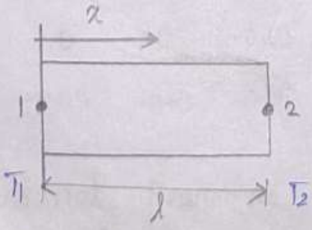
⇒ Radiation.

Heat Conduction is a mechanism of Heat Transfer from a region of high temperature to a region of low temperature within a medium. Eg. Heating of Steel rod.

Convection is a process of Heat Transfer that will occur between a solid surface and a fluid medium when they are at different temperature. Eg. Boiling of Coffee

Radiation is the heat transfer from one body to another body without any transmitting medium. Eg Solar cell

Derivation of Temperature function (T) and Shape function (N) for the One Dimensional Heat Conduction Element



Consider a bar element with Nodes 1 & Node 2.

T_1 & T_2 → Temperature at respective nodes.

T_1 and T_2 are considered as D.O.F of this bar element. Since the element has two D.O.F, it has two generalised co-ordinates.

$T = a_0 + a_1 x$

$a_0, a_1 \rightarrow$ Global (or) Generalised Co-Ordinates

At Node 1,

$T = T_1, \quad x = 0$

At Node 2,

$T = T_2, \quad x = l$

$T_1 = a_0$

$T_2 = a_0 + a_1 l$

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$ values in ①

$$T = \begin{bmatrix} 1 & x \end{bmatrix} \times \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \frac{1}{l} \begin{bmatrix} l-x & 0+x \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$T = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$T = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

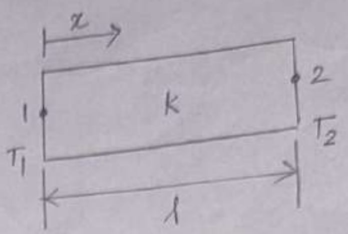
$$T = N_1 T_1 + N_2 T_2$$

where,

Shape functions, $N_1 = \frac{l-x}{l}$

$N_2 = \frac{x}{l}$

Derivation of Stiffness Matrix for One Dimensional Heat Conduction Element



where,

K → Thermal conductivity of the materials

T_1, T_2 → Temperature at the respective Nodes

W.K.T,

Stiffness Matrix,

$$[K] = \int_v [B]^T [D] [B] \, dv$$

In One dimensional element,

Temperature function,

$$T = N_1 T_1 + N_2 T_2$$

where,

$$N_1 = \frac{l-x}{l}$$

$$N_2 = \frac{x}{l}$$

W.K.T,

Strain Displacement Matrix,

$$[B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$$

$$[B]^T = \begin{Bmatrix} -1/l \\ 1/l \end{Bmatrix}$$

In One Dimensional Heat Conduction Problems,

$[D] = [k] = K$ = Thermal Conductivity

Sub $[B]$, $[B]^T$ and $[D]$ values in Stiffness Matrix eqn,

$$[K_c] = \int_0^l \begin{Bmatrix} -1/l \\ 1/l \end{Bmatrix} \times K \times \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dv$$

$$= \int_0^l \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} K \, dv$$

$$= \int_0^l \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} \times K \times A \, dx \qquad [\because dv = A\,dx]$$

$$[K_c] = AK \int_0^l \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} dx$$

$$= AK \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} [l - 0]$$

$$= \frac{AKl}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_c] = \frac{AK}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$A \rightarrow$ Area of the element, m^2

$K \rightarrow$ Thermal conductivity, W/mK

$l \rightarrow$ Length of the element, m

Finite Element Equation for One-Dimensional Heat Conduction Problems

$$\{F\} = [K_c]\{T\}$$

$\{F\}$ - Element Force Vector

$[K_c]$ - Stiffness Matrix

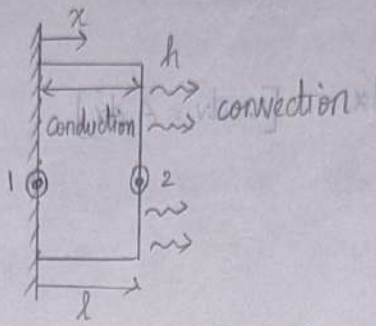
$\{T\}$ - Nodal Temperature

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = [K_c] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AK}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Case (i)

One Dimensional Heat conduction with free end convection



Consider a One Dimensional element with Nodes 1 and 2. T_1 and T_2 are the temperatures at the respective Nodes. Assume convection occurs from only right end of the element.

Stiffness matrix for the One Dimensional Heat conduction element is

$$[K_c] = \frac{AK}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The convection term contribution to the Stiffness matrix is

$$[K_h]_{end} = \iint_A h\ [N]^T [N]\ dA$$

where,

$h \rightarrow$ Heat Transfer Coefficient, W/m^2K

$N \rightarrow$ Shape factor

W.K.T,

$$[N] = [N_1\ \ N_2] = \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix}$$

At Node 2, $x = l$

$$[N] = [N_1\ \ N_2] = [0\ \ 1]$$

$$[N]^T = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Substitute $[N]$ and $[N]^T$ values

$$[K_h]_{end} = \iint_A h \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} dA$$

$$= h \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \int dA$$

$$= hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Stiffness matrix,

$$[K] = [K_c] + [K_h]$$

$$[K] = \frac{AK}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The convection from the free end

$$\{F_h\} = h\, T_\infty\, A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

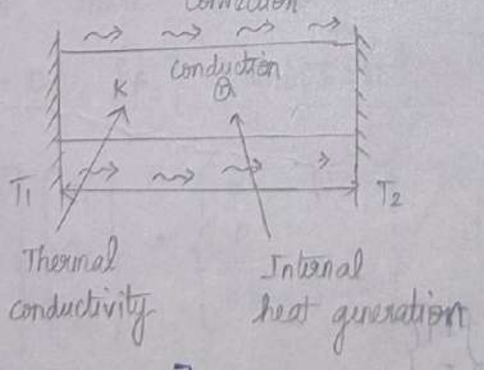
General eqn,

$$\{F\} = [K]\{T\}$$

$$h\, T_\infty\, A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \left[\frac{AK}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

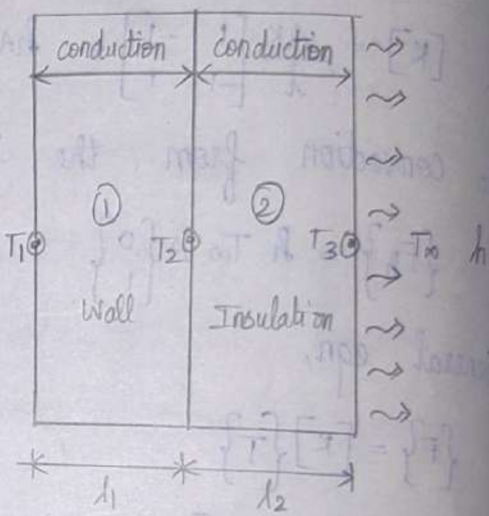
Case (ii)

One Dimensional element with conduction, convection and Internal heat generation



$$\left[\frac{AK}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPl}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{QAl + PhT_\infty l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

1. A wall of 0.6m thickness having thermal conductivity of 1.2 W/mK. The wall is to be insulated with a material of thickness 0.06m having an average thermal conductivity of 0.3 W/mK. The inner surface temperature is 1000°C and outside of the insulation is exposed to atmospheric air at 30°C with heat transfer coefficient of 35 W/m²K. Calculate the Nodal temperatures.



Given

$l_1 = 0.6m$, $l_2 = 0.06m$

$K_1 = 1.2 W/mK$, $K_2 = 0.3 W/mK$

$T_1 = 1000° + 273 = 1273K$

$T_\infty = 30°C + 273 = 303K$

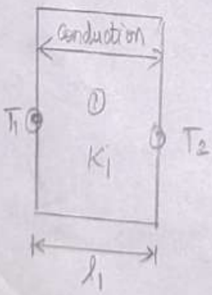
$h = 35 \ W/m^2K$

To find

Nodal temperatures T_2 & T_3

Solution

For element ① (Node 1, 2)



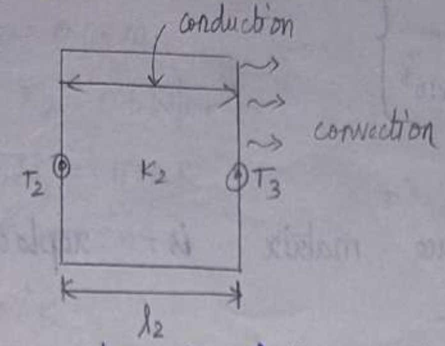
Finite Element Eqn,

$$\Rightarrow \frac{A_1K_1}{l_1}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\F_2\end{Bmatrix}$$

$$\Rightarrow \frac{1\times 1.2}{0.6}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\F_2\end{Bmatrix} \qquad [\because A = 1m^2]$$

$$\begin{Bmatrix}F_1\\F_2\end{Bmatrix} = \begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}\begin{Bmatrix}T_1\\T_2\end{Bmatrix} \quad - ①$$

For element ② (Node 2,3)



This element consists of both conduction and convection

So, the finite element eqn,

$$\left(\frac{A_2 K_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = h\,T_\infty A\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\left(\frac{0.3}{0.06}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + 35\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = 35 \times 303\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\begin{bmatrix}5 & -5\\ -5 & 40\end{bmatrix}\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10.605\times10^3\end{Bmatrix} \quad — ②$$

Assemble the eqns ① & ②

$$\begin{bmatrix}2 & -2 & 0\\ -2 & 7 & -5\\ 0 & -5 & 40\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\\ F_3\end{Bmatrix}$$

In this problem, there is no heat generation

$$\{F_1\} = \{F_2\} = 0 \;,\; \{F_3\} = 10.605\times10^3$$

$$\begin{bmatrix}2 & -2 & 0\\ -2 & 7 & -5\\ 0 & -5 & 40\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 10.605\times10^3\end{Bmatrix}$$

Solve the above Finite Element Eqn by using following steps

Step 1

The first row and first column of the stiffness matrix $[K]$ have been set equal to zero except for the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step 2

The first row of the force matrix is replaced by known temperature at Node 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1273 \\ 0 \\ 10.605 \times 10^3 \end{Bmatrix}$$

Step 3

The second row, first column of stiffness matrix [K] value is multiplied by known temperature at Node 1 $(-2 \times 1273 = -2546)$ and replace second force Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & -5 & 40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1273 \\ 2546 \\ 10.605 \times 10^3 \end{Bmatrix}$$

$$7T_2 - 5T_3 = 2546$$

$$-5T_2 + 40T_3 = 10.605 \times 10^3$$

Solving,

$$T_2 = 607.313 \text{ K}$$

$$T_3 = 341.03 \text{ K}$$

Tutorial

2. A wall of 0.8 m thickness having thermal conductivity of 1.4 W/mK. The wall is to be insulated with a material of thickness 0.08 m having an average thermal conductivity of 0.6 W/mK. The inner surface temperature is 900°c and outside of the insulation is exposed to atmospheric air at 26°c with heat transfer coefficient of 35 W/m²K. Calculate the Nodal temperature.

Given

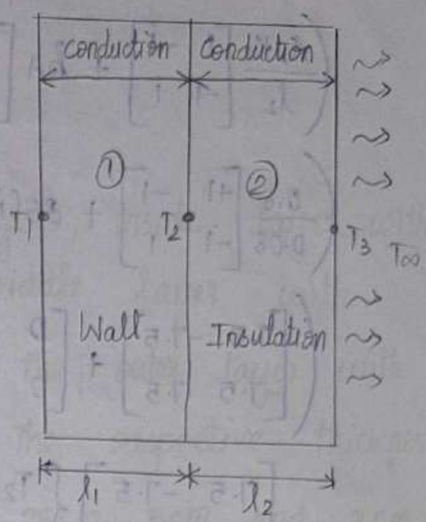
$l_1 = 0.8\,m$, $l_2 = 0.08\,m$

$K_1 = 1.4\,W/mK$, $K_2 = 0.6\,W/mK$

$T_1 = 900^\circ C + 273 = 1173\,K$

$T_\infty = 26^\circ C + 273 = 299\,K$

$h = 35\,W/m^2K$

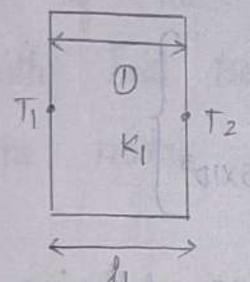


To find

Nodal temperatures T_2 & T_3

Solution

For element ① (Node 1, 2)



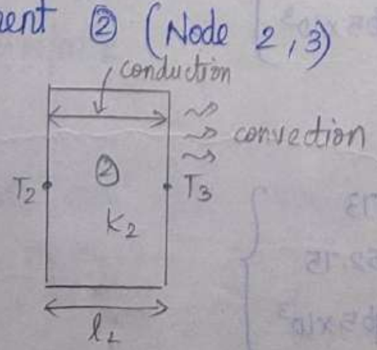
Finite element eqn,

$$\frac{A_1K_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

$$\frac{1 \times 1.4}{0.8}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

$$\begin{bmatrix}1.75 & -1.75\\ -1.75 & 1.75\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix} \quad \text{---} \ ①$$

For element ② (Node 2, 3)



This element consists of both conduction and convection

So, the Finite Element eqn,

$$\left(\frac{A_2K_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = hT_\infty A\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\left(\frac{0.6}{0.08}\begin{bmatrix}+1 & -1\\ -1 & 1\end{bmatrix} + 35(1)\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = 35 \times 299 \times 1\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\left(\begin{bmatrix}7.5 & -7.5\\ -7.5 & 7.5\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 35\end{bmatrix}\right)\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = 10.465 \times 10^3\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\begin{bmatrix}7.5 & -7.5\\ -7.5 & 42.5\end{bmatrix}\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10.465 \times 10^3\end{Bmatrix} \quad — \ (2)$$

Assemble the eqns (1) & (2)

$$\begin{bmatrix}1.75 & -1.75 & 0\\ -1.75 & 9.25 & -7.5\\ 0 & -7.5 & 42.5\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 10.465 \times 10^3\end{Bmatrix}$$

Solve the above finite element eqns using following steps

Step 1

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 9.25 & -7.5\\ 0 & -7.5 & 42.5\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 10.465 \times 10^3\end{Bmatrix}$$

Step 2

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 9.25 & -7.5\\ 0 & -7.5 & 42.5\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}1173\\ 0\\ 10.465 \times 10^3\end{Bmatrix}$$

Step 3

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 9.25 & -7.5\\ 0 & -7.5 & 42.5\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}1173\\ 2052.75\\ 10.465 \times 10^3\end{Bmatrix}$$

$$9.25T_2 - 7.5T_3 = 2052.75$$

$$-7.5T_2 + 42.5T_3 = 10.465 \times 10^3$$

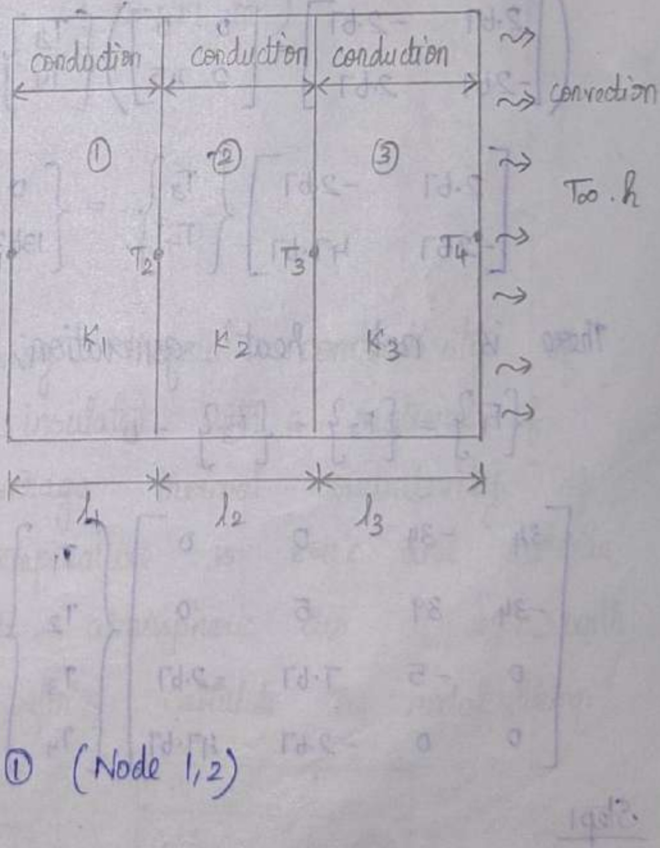
Solving,

$T_2 = 491.07\ K$

$T_3 = 333.89\ K$

3. A furnace was made up of three layers, inside layer with the thermal conductivity 8.5 W/mK, the middle layer with the thermal conductivity of 0.25 W/mK, the outer layer with the thermal conductivity of 0.08 W/mK. The respective thickness of the inner, middle, outer layers are 25cm, 5cm and 3cm respectively. The inside temperature of the wall is 600°C and outside of the wall is exposed to atmospheric air at 30°C with heat transfer coefficient of 45 W/m²K. Determine the nodal temperatures.

Given



$K_1 = 8.5\ W/mK$

$K_2 = 0.25\ W/mK$

$K_3 = 0.08\ W/mK$

$l_1 = 25cm = 0.25m$

$l_2 = 5cm = 0.05m$

$l_3 = 3cm = 0.03m$

$T_1 = 600°C = 873\ K$

$T_\infty = 30°C = 303\ K$

$h = 45\ W/m^2K$

Solution

Finite element eqn for element ① (Node 1,2)

$$\frac{A_1K_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\frac{1.85}{0.25}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\begin{bmatrix} 34 & -34 \\ -34 & 34 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Finite element eqn for element ② (Node 2,3)

$$\frac{A_2 K_2}{l_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

Finite element eqn for element ③ (Node 3,4)

This element is subjected to both conduction and convection

$$\left[\frac{A_3 K_3}{l_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = h T_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left(\frac{1 \times 0.08}{0.03} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 45 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = 45 \times 303 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\left(\begin{bmatrix} 2.67 & -2.67 \\ -2.67 & 2.67 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 45 \end{bmatrix} \right) \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 13635 \end{Bmatrix}$$

$$\begin{bmatrix} 2.67 & -2.67 \\ -2.67 & 47.67 \end{bmatrix} \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 13635 \end{Bmatrix}$$

There is not heat generation,

$$\{F_1\} = \{F_2\} = \{F_3\} = 0$$

$$\begin{bmatrix} 34 & -34 & 0 & 0 \\ -34 & 39 & 5 & 0 \\ 0 & -5 & 7.67 & -2.67 \\ 0 & 0 & -2.67 & 47.67 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 13635 \end{Bmatrix}$$

Step 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 39 & 5 & 0 \\ 0 & -5 & 7.67 & -2.67 \\ 0 & 0 & -2.67 & 47.67 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 13635 \end{Bmatrix}$$

Step 2

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 39 & 5 & 0 \\ 0 & -5 & 7.67 & -2.67 \\ 0 & 0 & -2.67 & 47.67 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 873 \\ 0 \\ 0 \\ 13635 \end{Bmatrix}$$

Step 3

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 39 & 5 & 0 \\ 0 & -5 & 7.67 & -2.67 \\ 0 & 0 & -2.67 & 47.67 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 873 \\ 29682 \\ 0 \\ 13635 \end{Bmatrix}$$

Using Gauss elimination method,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0.128 & 0 \\ 0 & 0 & 1 & -0.379 \\ 0 & 0 & 0 & 46.655 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 873 \\ 761.076 \\ 54.614 \\ 15.079 \times 10^3 \end{Bmatrix}$$

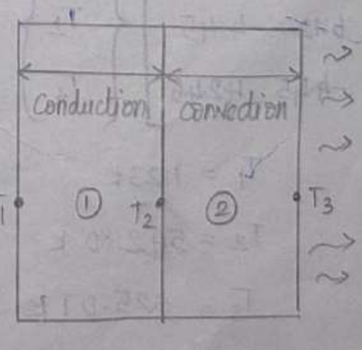
$T_4 = 323.21K$

$T_3 = 664.71K$

$T_2 = 846.08\ K$

4. A wall of 0.9 m thickness having thermal conductivity of 1.5 W/mK. The wall is to be insulated with a material of thickness 0.09 m having an average thermal conductivity of 0.4 W/mK. The inner surface temperature is 850°C and outside of the insulation is exposed to atmospheric air at 27°C with heat transfer coefficient of 38 W/m^2K. Calculate the nodal temp.

Given



$l_1 = 0.9\ m$

$K_1 = 1.5\ W/mK$

$l_2 = 0.09\ m$

$K_2 = 0.4\ W/mK$

$T_1 = 850 + 273 = 1123\ K$

$T_\infty = 27 + 273 = 300\ K$

Solution

For element ①,

$$\frac{A_1K_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

$$\frac{1\times 1.5}{0.9}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

$$\begin{bmatrix}1.67 & -1.67\\ -1.67 & 1.67\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

For element ②

$$\frac{A_2K_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = hT_\infty A\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\left(\frac{1\times 0.4}{0.09}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + 38\times 1\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = 38\times 300\times 1\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\begin{bmatrix}4.45 & -4.45\\ -4.45 & 42.45\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}0\\ 11.4\times 10^3\end{Bmatrix}$$

Assembling the above finite element eqn,

$$\begin{bmatrix}1.67 & -1.67 & 0\\ -1.67 & 6.12 & -4.45\\ 0 & -4.45 & 42.45\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 11.4\times 10^3\end{Bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ -1.67 & 6.12 & -4.45\\ 0 & -4.45 & 42.45\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}1123\\ 0\\ 11.4\times 10^3\end{Bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 6.12 & -4.45\\ 0 & -4.45 & 42.45\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}1123\\ 1875.41\\ 11.4\times 10^3\end{Bmatrix}$$

$T_1 = 1123\,K$

$T_2 = 542.80\,K$

$T_3 = 325.07\,K$

ISOPARAMETRIC ELEMENTS

Gaussian Quadrature and Applications to plane stress problem

Point Calculation $2n-1$ = Higher order

No. of points 'n'	Location	Corresponding weights W_i
1	$x_1 = 0.000....$	2.000
2	$x_1, x_2 = \pm\sqrt{1/3}$ $= \pm 0.577350269$	1.000
3	$x_1, x_3 = \pm\sqrt{3/5}$, $x_2 = 0.000$ $= \pm 0.774596669$	$\frac{5}{9} = 0.5555555$ $x_2 = \frac{8}{9} = 0.8888$
4	$x_1, x_4 = \pm 0.8611363116$ $x_2, x_3 = \pm 0.3399810436$	0.3478548451 0.6521451549

1. Evaluate $\int_{-1}^{1} (x^4 + x^2)\, dx$ by applying 3 point Gaussiun quadrature

Given

$$f(x) = x^4 + x^2$$

Integral $I = \int_{-1}^{1} (x^4 + x^2)\, dx$

$$x_1 = \sqrt{\frac{3}{5}} = 0.77459$$

$$x_2 = 0$$

$$x_3 = -\sqrt{\frac{3}{5}}$$

Weight,

$$W_1 = \frac{5}{9} = 0.5555$$

$$W_2 = \frac{8}{9} = 0.8888$$

$$W_3 = \frac{5}{9} = 0.5555$$

W.K.T,

$$f(x) = x^4 + x^2$$

$f(x_1) = x_1^4 + x_1^2$

$= (0.774596)^4 + (0.774596)^2$

$f(x_1) = 0.959997$

$W_1 f(x_1) = 0.5555 \times 0.959997$

$W_1 f(x_1) = 0.533278$

$f(x_2) = x_2^4 + x_2^2$

$f(x_2) = 0$

$W_2 f(x_2) = 0$

$f(x_3) = x_3^4 + x_3^2$

$= (-0.774596)^4 + (-0.774596)^2$

$f(x_3) = 0.959997$

$W_3 f(x_3) = 0.5555 \times 0.959997$

$W_3 f(x_3) = 0.533278$

$W_1 f(x_1) + W_2 f(x_2) + W_3 f(x_3) = 0.533278 + 0 + 0.533278$

$W_1 f(x_1) + W_2 f(x_2) + W_3 f(x_3) = 1.066$

<u>Verification</u>

$$\int_{-1}^{1} (x^4 + x^2)\, dx = \left[\frac{x^5}{5} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \left[\left(\frac{1}{5} + \frac{1}{3}\right) - \left(\frac{-1}{5} - \frac{1}{3}\right)\right]$$

$$= 0.533 + 0.533$$

$$\int_{-1}^{1} (x^4 + x^2)\, dx = 1.066$$

2. Evaluate $\int_{-1}^{1} (x^4 - 3x + 7)\, dx$

$2n - 1 = 4$

$2n = 5$

$n = \frac{5}{2}$

$n = 2.5$

$f(x) = x^4 - 3x + 7$

Integral $I = \int_{-1}^{1} (x^4 - 3x + 7)$

Take $n = 3$

$x_1 = 0.77459$

$x_2 = 0$

$x_3 = -0.77459$

Weight,

$W_1 = \frac{5}{9} = 0.5555$

$W_2 = \frac{8}{9} = 0.8888$

$W_3 = \frac{5}{9} = 0.5555$

~~W.K.T,~~

$f(x) = x^4 - 3x + 7$

$f(x_1) = x_1^4 - 3x_1 + 7$

$= (0.77459)^4 - 3(0.77459) + 7$

$f(x_1) = 5.03621$

$W_1 f(x_1) = 0.5555 \times 5.03621$

$W_1 f(x_1) = 2.7976$

$f(x_2) = 7$

$W_2 f(x_2) = 6.2216$

$f(x_3) = x_3^4 - 3x_3 + 7$

$= (-0.77459)^4 - 3(-0.77459) + 7$

$f(x_3) = 9.6837$

$W_3 f(x_3) = 0.5555 \times 9.6837$

$W_3 f(x_3) = 5.3793$

$W_1 f(x_1) + W_2 f(x_2) + W_3 f(x_3) = 2.7976 + 6 + 5.3793$

$= 14.398$

Verification

$$\int_{-1}^{1}(x^4-3x+7)\,dx = \left[\frac{x^5}{5}-\frac{3x^2}{2}+7x\right]_{-1}^{1}$$

$$= \left[\frac{1}{5}-\frac{3}{2}+7\right]-\left[\frac{-1}{5}-\frac{3}{2}+7\right]$$

$$= 5.7 + 8.7$$

$$= 14.4$$

3. Evaluate $\int_{-1}^{1} e^{-x}\,dx$ by applying 3 point Gaussian quadrature

$f(x) = e^{-x}$

$x_1 = 0.77459$

$x_2 = 0$

$x_3 = 0.77459$

Weight

$W_1 = \frac{5}{9} = 0.5555$

$W_2 = \frac{8}{9} = 0.8888$

$W_3 = \frac{5}{9} = 0.5555$

$f(x_1) = e^{-x_1}$

$= e^{-(0.77459)}$

$f(x_1) = 0.4608$

$W_1 f(x_1) = 0.5555 \times 0.4608$

$W_1 f(x_1) = 0.2560$

$f(x_2) = e^{-0} = 1$

$W_2 f(x_2) = 0.8888 \times 1$

$W_2 f(x_2) = 0.8888$

$f(x_3) = e^{-(0.77459)}$

$f(x_3) = 2.1697$

$W_3 f(x_3) = 0.5555 \times 2.1697$

$W_3 f(x_3) = 1.2052$

$W_1 f(x_1) + W_2 f(x_2) + W_3 f(x_3) = 0.2560 + 0.8888 + 1.2052$

$= 2.350$

Verification

$$\int_{-1}^{1} e^{-x}\,dx = \left[-e^{-x}\right]_{-1}^{1}$$

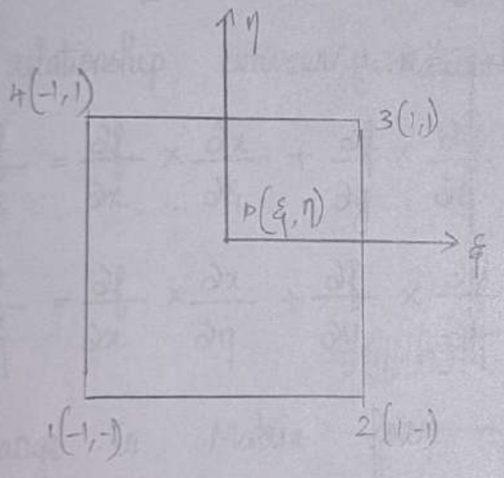
$$= (-e^{-1}) - (-e^{1})$$

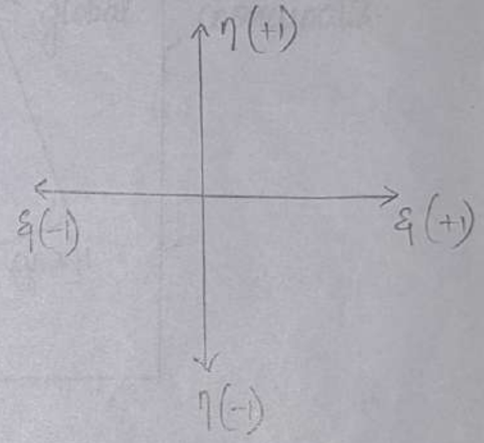
$$= -0.3678 + 2.7182$$

$$= 2.350$$

Isoparametric formulation

Shape function for 4 nodal rectangular parent element by using natural coordinate system and coordinate





At node 1 ; $\xi = -1, \ \eta = -1$

$N_1 = 1/4\,(1-\xi)(1-\eta)$

$N_2 = 1/4\,(1+\xi)(1-\eta)$

$N_3 = 1/4\,(1+\xi)(1+\eta)$

$N_4 = 1/4\,(1-\xi)(1+\eta)$

The displacement components

$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$

and

$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$

It can be written in matrix form as

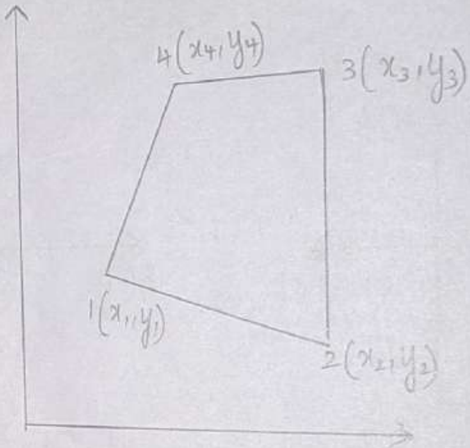
$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

In the isoparametric formulation,

i.e for global system the coordinate of the nodal points are (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4)

In order to mapping

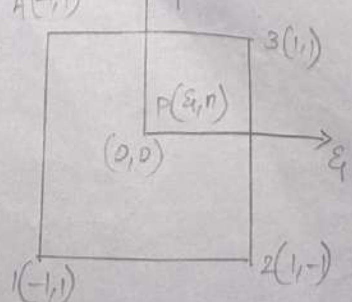
$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

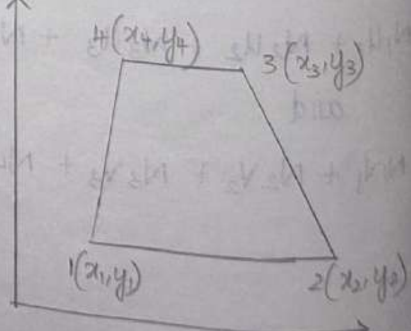


The above eqns can be written in Matrix form

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

<u>Element stiffness matrix equation for 4-nodal Isoparametric quadrilateral element</u>





The displacement function u for parent rectangular element, ...

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

the displacement function u for isoparametric elements,

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

Let $F = f(x, y)$

$$F = f\left[x(\xi, \eta), \, y(\xi, \eta)\right]$$

The relationship between natural and global coordinates

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \eta}$$

Arrange in Matrix form

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

$$= [J] \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

where $[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$

$$= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where, $J_{11} = \frac{\partial x}{\partial \xi}$, $J_{12} = \frac{\partial y}{\partial \xi}$

$J_{21} = \frac{\partial x}{\partial \eta}$, $J_{22} = \frac{\partial y}{\partial \eta}$

W.K.T,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$J_{11} = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4$$

Similarly,

J_{12}, J_{21}, J_{22}

W.K.T,

$$N_1 = 1/4 (1-\xi)(1-\eta)$$

$$N_2 = 1/4 (1+\xi)(1-\eta)$$

$$N_3 = 1/4 (1+\xi)(1+\eta)$$

$$N_4 = 1/4 (1-\xi)(1+\eta)$$

$$\frac{\partial N_1}{\partial \xi} = \frac{1}{4}(-1)(1-\eta) \qquad \frac{\partial N_1}{\partial \eta} = \frac{1}{4}(1-\xi)(-1)$$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1)(1-\eta) \qquad \frac{\partial N_2}{\partial \eta} = \frac{1}{4}(1+\xi)(-1)$$

$$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1)(1+\eta) \qquad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi)(1)$$

$$\frac{\partial N_4}{\partial \xi} = \frac{1}{4}(-1)(1+\eta) \qquad \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)(1)$$

Sub all the values in J_{11}, J_{12}, J_{21}, J_{22}

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1-\eta)x_4\right]$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{1}{4}\left[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

$$J_{22} = \frac{1}{4}\left[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\right]$$

W.K.T,

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$= \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

The strain displacement relation

$$e = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \partial v/\partial y \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

Substituting $f = u$

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

Illy

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

Strain, $\{e\} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & J_{12} \end{bmatrix} \begin{Bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \\ \partial v/\partial \xi \\ \partial v/\partial \eta \end{Bmatrix}$

W.K.T,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

Calculate $\begin{Bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \\ \partial v/\partial \xi \\ \partial v/\partial \eta \end{Bmatrix}$

$$\{e\} = [B]\{u\}$$

Strain displacement matrix,

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 \\ 0 & -(1-\eta) & 0 & -(1-\eta) & 0 & (1+\eta) \\ 0 & -(1+\xi) & 0 & -(1+\xi) & 0 & (1+\xi) \end{bmatrix}$$

For isoparametric

$$[K] = t \iint [B]^T [D][B]\, \partial x\, \partial y$$

For natural coordinates

$$[K] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B] \times |J| \times \partial \xi \times \partial \eta$$

where,

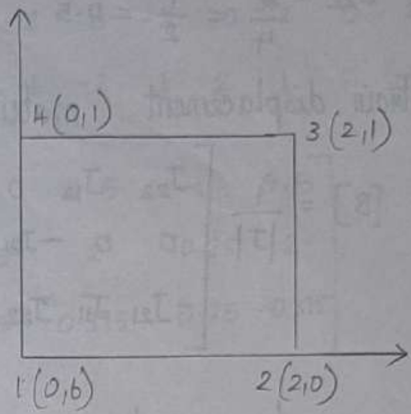
$$[D] = \frac{E}{(1-v^2)}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \quad \text{Plane stress}$$

$$= \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \quad \text{Plane strain}$$

1. A 4 noded rectangular element as shown in fig. Determine the following i) Jacobian Matrix ii) Strain displacement matrix iii) Element stress matrix. Take $E = 2 \times 10^5\ N/mm^2$, $\nu = 0.25$,

$u = [0, 0, 0.003, 0.004, 0.006, 0.004, 0, 0]$

$\xi = 0, \ \eta = 0$



To find

1. Jacobian matrix [J]
2. Strain displacement matrix [B]
3. Element stresses $[\sigma]$

Solution

$x_1 = 0, \ y_1 = 0$

$x_2 = 2, \ y_2 = 0$

$x_3 = 2, \ y_3 = 1$

$x_4 = 0, \ y_4 = 01$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$= \frac{1}{4}\left[-(1)0 + (1)2 + (1)2 - (1)0\right]$$

$$J_{11} = 1$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$= \frac{1}{4}\left[-(1)0 + (1)0 + (1)1 - (1)01\right]$$

$$J_{12} = \frac{1}{4}[0] = 0$$

$$J_{21} = \frac{1}{4}\left[-(1-\xi)x_1 (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

$$= \frac{1}{4}\left[-(1)0 - (1)2 + (1)2 + (1)0\right]$$

$$J_{21} = 0$$

$$J_{22} = \frac{1}{4}\left[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\right]$$

$$= \frac{1}{4}\left[-(1)0 - (1)0 + (1)1 + (1)1\right]$$

$$J_{22} = \frac{2}{4} = \frac{1}{2} = 0.5$$

Strain displacement matrix,

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1+\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix}$$

$$|J| = \begin{vmatrix} 1 & 0 \\ 0 & 0.5 \end{vmatrix}$$

$$= 0.5 - 0$$

$$|J| = 0.5$$

$$[B] = \frac{1}{0.5}\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0.5 & 0 \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} \quad '' \quad \end{bmatrix}$$

$$[B] = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

$$= \frac{2\times10^5}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

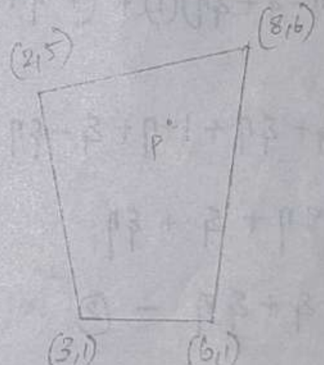
$$[D] = 213.33 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \times 10^3$$

$$[D][B] = \begin{matrix} 213.33 \\ \times 10^3 \end{matrix} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \times 0.25 \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$[D][B] = \begin{matrix} 53.3325 \\ \times 10^3 \end{matrix} \begin{bmatrix} -1 & -0.5 & 1 & -0.5 & 1 & 0.5 & -1 & 0.5 \\ -0.25 & -2 & 0.25 & -2 & 0.25 & 2 & -0.25 & 2 \\ -0.75 & -0.375 & -0.75 & 0.375 & 0.75 & 0.375 & 0.75 & -0.375 \end{bmatrix}$$

$$\{\sigma\} = [D][B]\{u\}$$

2. For the isoparametric quadrilateral element shown in fig. Determine the local coordinates of the point P which has Cartesian coordinates (7,4)



To find

Local coordinates (ξ, η)

Solution

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_5 = \frac{1}{4}(1-\xi)(1+\eta)$$

From diagram,

$x_1 = 3,\ y_1 = 1$

$x_2 = 6,\ y_2 = 1$

$x_3 = 8,\ y_3 = 6$

$x_4 = 2,\ y_4 = 5$

$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

$7 = N_1 3 + N_2 6 + N_3 8 + N_4 2$

$7 = \left[\left(\frac{1}{4}(1-\xi)(1-\eta)\right)(3) + \left(\frac{1}{4}(1+\xi)(1-\eta)\right)(6) + \left(\frac{1}{4}(1+\xi)(1+\eta)\right)(8) + \left(\frac{1}{4}(1-\xi)(1+\eta)\right)\right.$

$7 = \frac{1}{4}\left[(1-\eta-\xi+\xi\eta)(3) + (1-\eta+\xi-\xi\eta)(6) + (1+\eta+\xi+\xi\eta)(8) + (1+\eta-\xi-\xi\eta)(2)\right]$

$28 = 3-3\eta-3\xi+3\xi\eta+6-6\eta+6\xi-6\xi\eta+8+8\eta+8\xi+8\xi\eta+2+2\eta-2\xi-2\xi\eta$

$28 = 19+\eta+9\xi+3\eta\xi$

$9 = \eta+9\xi+3\eta\xi \quad - \quad (1)$

$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$

$4 = N_1(1) + N_2(1) + N_3(6) + N_4(5)$

$4 = \left[\left(\frac{1}{4}(1-\xi)(1-\eta)\right)(1) + \left(\frac{1}{4}(1+\xi)(1-\eta)\right)(1) + \left(\frac{1}{4}(1+\xi)(1+\eta)\right)(6) + \left(\frac{1}{4}(1-\xi)(1+\eta)\right)(5)\right]$

$4 = \frac{1}{4}\left[(1-\eta-\xi+\xi\eta)(1) + (1-\eta+\xi-\xi\eta)(1) + (1+\eta+\xi+\xi\eta)(6) + (1+\eta-\xi-\xi\eta)(5)\right.$

$16 = 1-\eta-\xi+\xi\eta+1-\eta+\xi-\xi\eta+6+6\eta+6\xi+6\xi\eta+5+5\eta-5\xi-5\xi\eta$

$16 = 13+9\eta+\xi+\xi\eta$

$3 = 9\eta+\xi+\xi\eta \quad - \quad (2)$

Equating (1) & (2)

$(1) \Rightarrow 9 = \eta+9\xi+3\eta\xi$

$(2)\times 3 \Rightarrow 9 = 27\eta+3\xi+3\eta\xi$

$0 = -26\eta+6\xi$

$6\xi = 26\eta$

$\xi = \frac{26}{6}\eta$

$\xi = 4.333\,\eta$

Sub $\xi = 4.333\eta$ in ②

$$3 = 9\eta + 4.333\eta + 4.333\eta^2$$

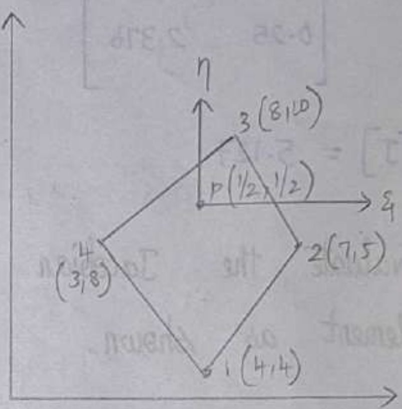
$$3 = 13.33\eta + 4.333\eta^2$$

$$\eta = 0.2\ ,\ -3.28$$

$$\xi = 4.333 \times 0.2$$

$$\xi = 0.866$$

3. Evaluate A, J at $\xi = \eta = \frac{1}{2}$ for the linear quadrilateral element as shown



Co-Ordinates

$x_1 = 4,\ y_1 = 4$

$x_2 = 7,\ y_2 = 5$

$x_3 = 8,\ y_3 = 10$

$x_4 = 3,\ y_4 = 8$

$x = \frac{1}{2},\ y = \frac{1}{2}$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$= \frac{1}{4}\left[-(1-0.5)4 + (1-0.5)7 + (1+0.5)8 - (1+0.5)3\right]$$

$$J_{11} = 2.25$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$= \frac{1}{4}\left[-(1-0.5)4 + (1-0.5)5 + (1+0.5)10 - (1+0.5)8\right]$$

$$J_{12} = 0.875$$

$$J_{21} = \frac{1}{4}\left[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

$$= \frac{1}{4}\left[-(1-0.5) - (1+0.5)7 + (1+0.5)8 + (1-0.5)3\right]$$

$$= 0.25$$

$$J_{22} = \frac{1}{4}\left[-(1-\xi)y_1 + (1+\xi)y_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

$$= \frac{1}{4}\left[-(1-0.5)4 + (1+0.5)5 + (1+0.5)10 + (1-\xi)8\right]$$

$$= 2.376$$

$$[J] = \begin{bmatrix} 2.25 & 0.875 \\ 0.25 & 2.376 \end{bmatrix}$$

$$[J] = 5.127$$

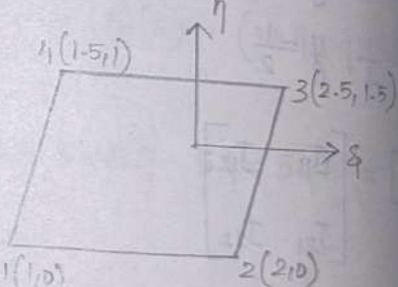
4. Evaluate the Jacobian matrix for the isoparametric quadrilateral element as shown.

$x_1 = 1,\ y_1 = 0$

$x_2 = 2,\ y_2 = 0$

$x_3 = 2.5,\ y_3 = 1.5$

$x_4 = 1.5,\ y_4 = 1$



Shape function,

$$N_1 = \frac{1}{4}\left[(1-\xi)(1-\eta)\right]$$

$$N_2 = \frac{1}{4}\left[(1+\xi)(1-\eta)\right]$$

$$N_3 = \frac{1}{4}\left[(1+\xi)(1+\eta)\right]$$

$$N_4 = \frac{1}{4}\left[(1-\xi)(1+\eta)\right]$$

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$x = \frac{1}{4}\left[1(1-\xi)(1-\eta) + 2(1+\xi)(1-\eta) + 2.5(1+\xi)(1+\eta) + 1.5(1-\xi)(1+\eta)\right]$$

$$y = \frac{1}{4}\left[0 + 0 + 1.5(1+\xi)(1+\eta) + 1(1-\xi)(1+\eta)\right]$$

$$= \frac{1}{4}\left[1.5 + 1.5\eta + 1.5\xi + 1.5\xi\eta + 1 + \eta - \xi - \xi\eta\right]$$

$$= \frac{1}{4}\left[2.5 + 2\eta + 0.5\xi + 0.5\xi\eta\right]$$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{1}{4}\left[0 + 0 + 2\right] = \frac{1}{2} = 0.5$$

$$J_{12} = \frac{\partial y}{\partial \xi} = \frac{1}{4}\left[0 + 0 + 0.5 + 0.5\eta\right] = \frac{0.5}{4}\left[1+\eta\right]$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{1}{4}(1) = \frac{1}{4}$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \frac{1}{4}\left[2.5 + 0.5\xi\right] = 0.125\left[5+\xi\right]$$

$$[J] = \begin{bmatrix} 0.5 & \frac{0.5}{4}(1+\eta) \\ \frac{1}{4} & 0.125(5+\xi) \end{bmatrix}$$