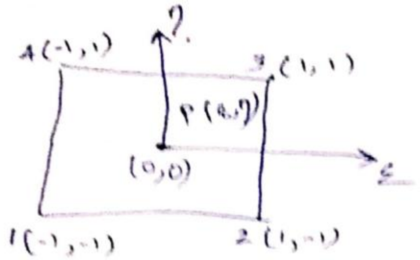
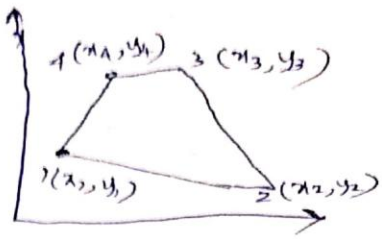
Element stiffness matrix Equation for 4 Noded Isoparametric Quadrilateral Element :



• Parent element



Isoparametric quadrilateral element

The displacement function u for parent rectangular element,

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

The displacement function u for isoparametric quadrilateral element,

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

Let $f = f(x, y)$

$$f = f\left[x(\xi,\eta),\ y(\xi,\eta)\right]$$

The relationship between natural & global Co-ordinates can be calculated by using chain rule of partial differential equation,

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \eta}$$

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Arranging the above equation

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

$$= [J] \begin{Bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{Bmatrix}$$

where $[J]$ is the Jacobian matrix,

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \partial y/\partial \eta \end{bmatrix} \Rightarrow [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where,

$$J_{11} = \frac{\partial x}{\partial \xi}, \quad J_{12} = \frac{\partial y}{\partial \xi}$$

$$J_{21} = \partial x/\partial \eta, \quad J_{22} = \partial y/\partial \eta$$

WKT,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$J_{11} = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4 \rightarrow (1)$$

$$J_{12} = \frac{\partial N_1}{\partial \xi} y_1 + \frac{\partial N_2}{\partial \xi} y_2 + \frac{\partial N_3}{\partial \xi} y_3 + \frac{\partial N_4}{\partial \xi} y_4 \rightarrow (2)$$

$$J_{21} = \frac{\partial N_1}{\partial \eta} x_1 + \frac{\partial N_2}{\partial \eta} x_2 + \frac{\partial N_3}{\partial \eta} x_3 + \frac{\partial N_4}{\partial \eta} x_4 \rightarrow (3)$$

$$J_{22} = \frac{\partial N_1}{\partial \eta} y_1 + \frac{\partial N_2}{\partial \eta} y_2 + \frac{\partial N_3}{\partial \eta} y_3 + \frac{\partial N_4}{\partial \eta} y_4 \rightarrow (4)$$

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WKT,

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$\frac{\partial N_1}{\partial \xi} = \frac{1}{4}(-1)(1-\eta), \quad \frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1)(1-\eta)$$

$$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1)(1+\eta), \quad \partial N_4/\partial \xi = \frac{1}{4}(-1)(1+\eta)$$

$$\frac{\partial N_1}{\partial \eta} = \frac{1}{4}(1-\xi)(-1), \quad \frac{\partial N_2}{\partial \eta} = \frac{1}{4}(1+\xi)(-1)$$

$$\frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi)(1), \quad \partial N_4/\partial \eta = \frac{1}{4}(1-\xi)(1)$$

substitute all the values in equ ①, ②, ③ & ④

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{1}{4}\left[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

$$J_{22} = \frac{1}{4}\left[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\right]$$

WKT,

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

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$$\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$= \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}$$

The strain displacement relations,

$$e = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}$$

substituting $f = u$

$$= \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \partial u/\partial y \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \partial u/\partial \eta \end{Bmatrix}$$

$$lly$$

$$\begin{Bmatrix} \partial v/\partial x \\ \partial v/\partial y \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \partial v/\partial \xi \\ \partial v/\partial \eta \end{Bmatrix}$$

$$\text{Strain } \{e\} = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \begin{Bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \\ \partial v/\partial \xi \\ \partial v/\partial \eta \end{Bmatrix}$$

wkt,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4.$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

. calculate $\begin{Bmatrix} \partial u/\partial \varepsilon \\ \partial u/\partial \eta \\ \partial v/\partial \varepsilon \\ \partial v/\partial \eta \end{Bmatrix}$.

$$\{e\} = [B]\{u\}$$

Strain-displacement matrix [B]

$$= \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}$$

$$\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

For Isoparametric quadrilateral element

$$[K] = t \iint [B]^T [D] [B] \, \partial x \, \partial y$$

Natural co-ordinates,

$$[K] = t \int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] \times |J| \times \partial \varepsilon \times \partial \eta.$$

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$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$ Plane stress

$$= \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$ for plane strain.

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