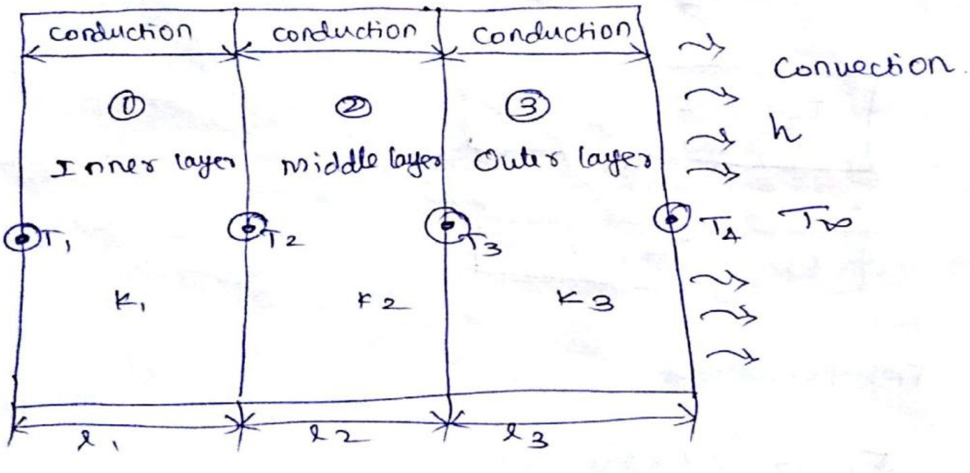
Problem ②

A furnace wall is made up of three layers, inside layer with thermal conductivity $\overset{9.8}{8.5}$ W/mK, the middle layer with conductivity $\overset{0.3}{0.25}$ W/mK, the outer layer with conductivity $\overset{0.09}{0.08}$ W/mK. The respective thickness of the inner, middle and outer layer are $\overset{27}{25}$ cm, $\overset{7}{5}$ cm and $\overset{5}{3}$ cm respectively. The inside temperature of the wall is $\overset{650^\circ C}{600^\circ C}$ and outside of the wall is exposed to atmospheric air at $\overset{32^\circ C}{30^\circ C}$ with heat transfer coefficient of $\overset{48}{45}$ W/m^{2}K. Determine the nodal temperatures.



Given:

$k_1 = 8.5$ W/mK, $k_2 = 0.25$ W/mK, $k_3 = 0.08$ W/mK

$l_1 = 25$ cm $= 0.25$ m, $l_2 = 5$ cm $= 0.05$ m, $l_3 = 3$ cm $= 0.03$ m.

$T_1 = 600^\circ C + 273 = 873$ K

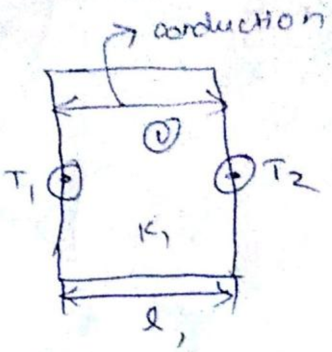
$T_\infty = 30^\circ C + 273 = 303$ K.

$h = 45$ W/m^{2}K.

To Find: Nodal temperatures (T_2, T_3 & T_4).

Solution :

For element 1 : (Nodes 1, 2) :



Finite element equation is

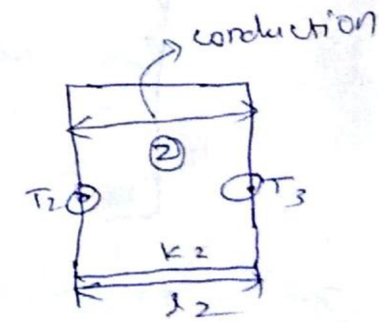
$$\frac{A_1 K_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

For unit area $A_1 = 1\,m^2$

$$\frac{8.5}{0.25}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

$$\begin{array}{cc} 1 & 2 \\ \left[\begin{matrix}34 & -34\\ -34 & 34\end{matrix}\right] & \begin{matrix}1\\ 2\end{matrix}\end{array}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

For element 2 : Nodes (2, 3).



Finite element equation is

$$\frac{A_2 K_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}F_2\\ F_3\end{Bmatrix}$$

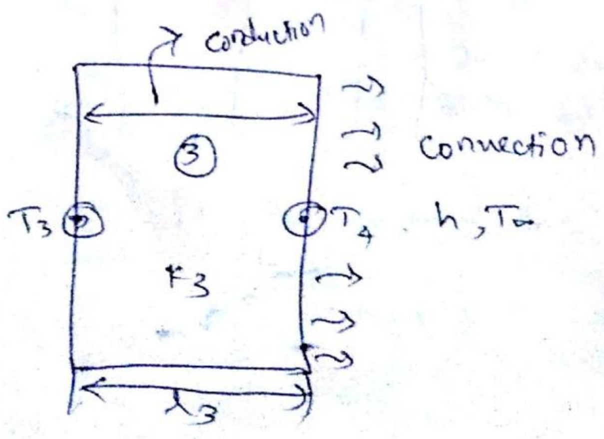
$$\frac{0.25}{0.05}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}F_2\\ F_3\end{Bmatrix}$$

$$\begin{array}{cc} 2 \quad 3 & \\ \left[\begin{matrix}5 & -5\\ -5 & 5\end{matrix}\right] & \begin{matrix}2\\ 3\end{matrix}\end{array}\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}F_2\\ F_3\end{Bmatrix}$$

For element 3 : (Nodes 3, 4)

This element is subjected to both conduction and convection,

So the finite element equation is



$$\left(\frac{A_3 K_3}{l_3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}T_3\\ T_4\end{Bmatrix} = h\,T_\infty A\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\left(\frac{0.08}{0.03}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + 45\times 1\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}T_3\\ T_4\end{Bmatrix} = 45\times 303\times 1\times\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

$$\begin{array}{ccc} & 3 & 4 & \\ & \begin{bmatrix} 2.666 & -2.666 \\ -2.666 & 47.666 \end{bmatrix} & \begin{array}{c}3\\4\end{array}\end{array}\begin{Bmatrix}T_3\\ T_4\end{Bmatrix} = \begin{Bmatrix}0\\ 13.635\times 10^3\end{Bmatrix}$$

Assemble the finite elements,

$$\begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 34 & -34 & 0 & 0 \\ -34 & 34+5 & -5 & 0 \\ 0 & -5 & 5+2.666 & -2.666 \\ 0 & 0 & -2.666 & 47.66 \end{bmatrix} & \begin{array}{c}1\\2\\3\\4\end{array} \end{array}\begin{Bmatrix}T_1\\ T_2\\ T_3\\ T_4\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\\ F_3\\ F_4\end{Bmatrix}$$

In this problem, there is no heat generation,

$$\{F_1\} = \{F_2\} = \{F_3\} = 0.$$

$$\{F_4\} = 13.635\times 10^3$$

$$\begin{bmatrix} 34 & -34 & 0 & 0 \\ -34 & 39 & -5 & 0 \\ 0 & -5 & 7.66 & -2.66 \\ 0 & 0 & -2.66 & 47.66 \end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\\ T_4\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\\ F_3\\ F_4\end{Bmatrix}$$

To solve the above equation, the following steps to be followed.

Step 1 :- The first row and first column of the stiffness matrix [K] have been set equal to 0 except for the main diagonal, which has been set equal to 1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 39 & -5 & 0 \\ 0 & -5 & 7.66 & -2.66 \\ 0 & 0 & -2.66 & 47.66 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 13.635\times10^3 \end{Bmatrix}$$

Step 2 : The first row of the force matrix is replaced by the known temp at node 1,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 39 & -5 & 0 \\ 0 & -5 & 7.66 & -2.66 \\ 0 & 0 & -2.66 & 47.66 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 873 \\ 0 \\ 0 \\ 13.635\times10^3 \end{Bmatrix}$$

Step 3 :- $-34\times873 = -29682 \rightarrow$ second row of the force matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 39 & -5 & 0 \\ 0 & -5 & 7.66 & -2.66 \\ 0 & 0 & -2.66 & 47.66 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 873 \\ 29682 \\ 0 \\ 13.635\times10^3 \end{Bmatrix}$$

Use Gaussian elimination method,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -0.122 & 0 \\ 0 & 0 & 1 & -0.379 \\ 0 & 0 & 0 & 46.615 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 873 \\ 761.076 \\ 541.614 \\ 15.079\times10^3 \end{Bmatrix}$$

$T_4 = 323.21\ K$

$T_3 = 664.11\ K$

$T_2 = 846.08\ K$ //.

verification:

Heat flow through composite wall,

$$Q = \frac{\Delta T_{overall}}{R}.$$

where $\Delta T = T_1 - T_\infty$

$$R = \frac{1}{\frac{1}{h_{in} A} + \frac{l_1}{K_1 A} + \frac{l_2}{K_2 A} + \frac{l_3}{k_3 A} + \frac{1}{h_{out} A}}$$

$$Q = \frac{873 - 303}{\frac{0.25}{8.5} + \frac{0.05}{0.25} + \frac{0.03}{0.08} + \frac{1}{45}}$$

$$Q = 909.62\ W/m^2$$

wkt,
$$Q = \frac{T_1 - T_\infty}{R} = \frac{T_1 - T_2}{R_1} = \frac{T_2 - T_3}{R_2} = \frac{T_3 - T_4}{R_3} = \frac{T_4 - T_\infty}{R_{outer}}$$

$$Q = \frac{T_1 - T_2}{\frac{l_1}{K_1 A}}$$

$$909.62 = \frac{873 - T_2}{\frac{0.25}{8.5}}$$

$$T_2 = 846.24\ K$$

$$(6) \Rightarrow Q = \frac{T_2 - T_3}{R_2}$$

$$= \frac{T_2 - T_3}{l_2 / K_2 A}$$

$$909.64 = \frac{846.24 - T_3}{\frac{0.05}{0.25}}$$

$$T_3 = 664.31\ K.$$

$$Q = \frac{T_3 - T_4}{\frac{l_3}{K_3 A}}$$

$$909.62 = \frac{664.31 - T_4}{\frac{0.03}{0.08}}$$

$$\boxed{T_4 = 323.20\ K}$$