Logical games: play objects and rules

Niraj Khare
Carnegie Mellon University Qatar

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Agenda

- Conditional statement and its truth value
- Proof techniques for conditional statements

Is the conclusion right?

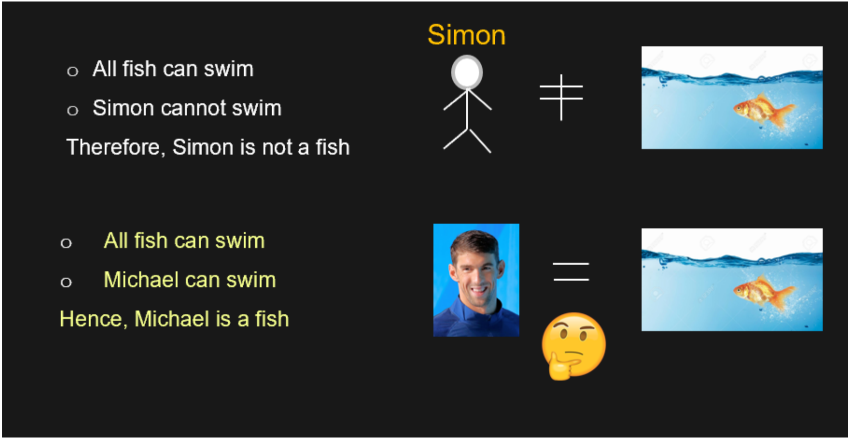

Figure: Is the conclusion right?

Is it true?

- If Tom is a parrot, then Tom is a bird..

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- For any real number x, if $x > 2$, then $x^2 > 4$.

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Definitions

Recall that natural numbers are counting numbers $\{1, 2, 3, \cdots\}$. The set of integers, denoted by $\mathbb{Z}$, is the set consisting of counting numbers along with zero and the negative values of the counting numbers. That is $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.

Definition

An integer n is called even if there exists an integer k such that $n = 2k$.

Definition

An integer n is called odd if there exists an integer k such that $n = 2k + 1$.

Remark

An integer is even or odd but not both.

Motivation I

For any integer n, show that $(7n^2 - 3n)^{2023}$ is an even integer.

Motivation II

In how many ways can 30 candies be distributed among 7 kids such that each kid gets an odd number of candies?

Motivation III

Exercise

For any integer n, show that if n is odd, then $3n^2 + 7$ is even.

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For any integer n, show that if $3n^2 + 7$ is even, then n is odd.

There are only three!

To prove conditional statements of the form $'p \to q'$, there are three techniques:

- Direct Proof
- Proof by contraposition (using contrapositive)
- Proof by contradiction

Some components of logical games

- Propositions (atomic and compound)
- connectors

Some examples of propositions

- $2 + 2 = 4$
- $2 + 2 = 7$
- The water is liquid at 25 degree celcius temperature and 1 bar pressure .
- A parrot is a bird.
- A monkey can fly.
- A donkey can solve an algebraic equation.

Definition of a proposition

Definition

A proposition is a declaration that uses only defined (i.e., without ambiguity) terms and has a unique truth value (i.e., either true or false but not both).

More Examples of propositions

- Karak tea is not available at Carnegie Mellon University Qatar on Feb 13, 2023.
- Roger Federer, as of February 2023, has more Wimbledon titles than any of his fellow players.
- Playstation 5 was launched in 2020.

These are not propositions.

- $x + 2 = 1$.
- Ahmed is three years old.
- Please sit down.
- Go and play outside.
- Is the elephant the largest land animal on earth?
- Help! I don't know the answer.

An interesting question

Determine whether the following sentence is a proposition or not?

- This sentence is false.

Compound Propositions

What's consistent across all the statements we have seen? Are there some limitations?

- The complexity arises when a proposition consists of multiple declaration, a.k.a compound propositions.

Example

- *John lives in London and Kate lives in France.*
- *Either Tom Cruise lives in the United States or he likes jumping off airplanes.*
- *If Tom is a parrot, then Tom is a bird.*

Symbolic representation of propositions

We can assign symbols to atomic propositions and use connectors such as ‘and’ and ‘or’ denoted by $\wedge$ and $\vee$ respectively to write compound propositions in symbolic form.

Let p denote the proposition ‘Tom Cruise lives in the United States’ and q denote ‘Tom Cruise likes jumping off airplanes’. Thus, we can denote the proposition:
‘Tom Cruise lives in the United States and he likes jumping off airplanes’ by

$p \wedge q$

Another example

How will you write in symbols: 'Either Tom Cruise lives in the United States or he likes jumping off airplanes' ?
We use old definitions of p and q.

$p \vee q$

Truth value of a compound proposition I

Note that $p \wedge q$ is 'true' when both p and q are true but 'false' if any of p or q is false.
pause

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth value of a compound proposition II

Note that $p \vee q$ is 'true' when at least one of p or q is true but 'false' if both of p and q are false.
pause

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Check your understanding

Assume the following propositions to be true.

- ‘Tom is a CS major’.
- ‘Tom loves plying video games’.
- ‘Tom doesn’t know programming’.

What is the truth value of each of the following propositions?

- Tom is a CS major and he knows programming.
- Tom is a CS major or he knows programming.
- Either Tom loves playing video games or he is a CS major.
- Tom loves playing video games and he is not a CS major.

Negation

For a proposition p, negation of p is denoted by $\neg p$.

For example, let p be the proposition 'Tom is a CS major'. The negation of the above proposition, $\neg p$, can be written in various ways in English. The following two ways are:

- It is not the case that Tom is a CS major.
- Tom is not a CS major.

Truth value of a negation

Whenever p is True, $\neg p$ is False and vice-versa.

p	$\neg p$
T	F
F	T

Truth value of a compound proposition with many connectors

Let p and q be propositions. What is the truth value of $(p \wedge q) \vee (\neg p)$?

p	q	$\neg p$	$p \wedge q$	$(p \wedge q) \vee (\neg p)$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

Topics covered

- What is an implication/conditional proposition?
- How to write symbolically a conditional proposition (a.k.a, statement) ?
- What is the truth value of a conditional statement?
- Different ways of writing a conditional statement in English.

Implication statement

Example

Figure: Cutting the branch implies fall

If $\cdots$, then $\cdots$.

Assume the guy in the previous photo is Tom.

If Tom is able to cut the branch completely, then Tom is going to fall.

Some examples of conditional propositions

- If Tom is a parrot, then Tom is a bird.
- If $2 + 2 = 7$, then dogs can do calculus.
- If you wash your boss's car, then you are promoted.
- If there is quiz today, then Ahmed is present in the class.
- If Ahmed is present in the class, then there is a quiz today.

Some examples of conditional propositions

- If a donkey can solve an algebraic equation, then dogs can excel in Calculus.
- If a person is the president, then he/she is at least 35 years old.
- If you are promoted, then you must have washed your boss' car.
- If a person is at least 35 years old, then he/she is the president.
- If Tom is a bird, then Tom is a parrot.

Truth value of a conditional statement

If you are inside the CMUQ building by 6 AM (Doha time) on Aug 1, 2022, then you will be hired by Google (by the evening of the same day).

Suppose you fulfill the hypothesis but Google doesn't hire you by the evening. It means the above claim is wrong.

Thus, a conditional statement is false when the hypothesis is true (i.e., holds) but the conclusion fails to occur (i.e., the conclusion is false). This is the only situation when a conditional statement is false.

Symbolic representation of a conditional statement

For a hypothesis (which is a proposition) p and a conclusion (which is a proposition as well) q, the implication: ‘If p, then q’ is denoted by $p \to q$.

Definition of $p \to q$ (a.k.a, its truth table)

The proposition $p \to q$ is 'False' when p is 'True' and q is 'False'. The proposition $p \to q$ is 'True' otherwise.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Do we understand the definition?

Determine whether the following propositions are ‘True’ or ‘False’?

- If $2 + 2 = 7$, then a cat is bigger than an elephant.
- If $2 + 2 = 7$, then $2 + 2 = 4$.
- If $2 + 2 = 4$, then $2 + 2 = 7$.
- If $2 + 2 = 7$, then a camel can live without water for more than two days.

$p \to q \equiv ((\neg q) \to (\neg p))$

For any propositions p and q, the proposition $(\neq q) \to (\neg p)$ is called the contrapositive of $p \to q$. Note that
$p \to q \equiv ((\neg q) \to (\neg p))$.

p	q	$p \to q$	$\neg p$	$\neg q$	$(\neg q) \to (\neg p)$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Equivalent implications using contrapositive

The following implications are equivalent as these statements are the contrapositives of each other. Note that the contrapositive of the implication $(\neg q) \to (\neg p)$ is the implication $p \to q$.

Example

- *If Tom is a parrot, then Tom is a bird.*
- *If Tom is not a bird, then Tom is not a parrot.*

$\neg(p \to q) \equiv (p \wedge (\neg q))$

Why $\neg(p \to q) \equiv (p \wedge (\neg q))$?

Some examples for direct Proof

Prove the following statements:

- For any two even integers, their sum is even.
- The product of any two odd integers is an odd integer.
- For any integer n , if n is odd then n^2 is odd.
- For any integer n , if n is odd, then $n = 4m + 1$ or $n = 4m + 3$ for some integer m.
- For any real numbers x and y, $x + y = \max\{x, y\} + \min\{x, y\}$.

An example of a direct proof

For any integer n , if n is odd then n^2 is odd.

Proof.

As n is odd, by definition of odd there exists an integer k such that $n = 2k + 1$.Thue,

$$\begin{aligned} \Rightarrow n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

As k is an integer and the sum and the product of any two integers is an integer, $2k^2 + 2k$ is an integer. Therefore, $n^2 = 2(2k^2 + 2k) + 1 = 2m + 1$ where $m = 2k^2 + 2k$ is an integer. By definition of odd, n^2 is an odd integer. □

Closure property of integers

Remark

Note that the sum and the product of any two integers is an integer is know as the closure property.

Some examples for proof by contraposition

- For any two integers m and n, if mn is even , then at least one of the integers m or n is even.
- For any integer n, if n^2 is odd , then n is odd.
- For any integer n, if n^2 is even , then n is even.
- For any integer n, if $7n^2 + 10$ is odd , then n is odd.

An example of proof by contraposition

For any integer n, if $3n^2 + 7$ is even , then n is odd.

Proof.

Consider the contrapositive: "For any integer n, if n in not odd (i.e., even), then $3n^2 + 7$ is not even (i.e., odd)." As the truth value of the contrapositive is the same as the original implication, it's enough to show that the contrapositive is true. As n is even, $n = 2k$ for some integer k by definition of even.

$$\begin{aligned} \Rightarrow 3n^2 + 7 &= 3(2k)^2 + 7 \\ &= 12k^2 + 6 + 1 \\ &= 2(6k^2 + 3) + 1. \end{aligned}$$

As k is an integer by closure property of integers $6k^2 + 3$ is an integer. Therefore, $3n^2 + 7 = 2(m) + 1$ is odd as $m = 6k^2 + 3$ is an integer. □

Rational Number

Definition

A real number x is called a rational number if there exist integers p and q such that $q \neq 0$ and $x = \frac{p}{q}$.

Example

- $\frac{3}{-2}$.
- $-3 = \frac{3}{1}$.
- $0 = \frac{0}{1}$.
- $4.723\overline{723}\cdots = \frac{4719}{999}$.

Proof by contradiction: Island Puzzle

Knights always speak the truth. Knaves always tell a lie. Knights and knaves dress up the same way (and by looking at them, you cannot tell which tribes they belong to).

A : B is a knight.

B : We are from the different tribes.

Proof by contradiction: more problems

Let $\mathbb{Q}$ denote the set of rational numbers and Irr denote the set of irrational numbers.

- There exists an irrational number (i.e., $\sqrt{2}$ is an irrational number).
- For all reals x, $((x \in Irr) \rightarrow (2x \in Irr))$.
- For all reals x and y, $(((x \in Irr) \wedge ((y \in \mathbb{Q}) \wedge (y \neq 0))) \rightarrow (xy \in Irr))$.
- For all reals x and y, $((x \in Irr) \wedge (y \in \mathbb{Q}) \rightarrow (x + y \in Irr))$.

Questions

- For any integer n, show that $(7n^2 - 3n)^{2023}$ is an even integer.
- For any positive integer n, show that any selection S of $n+1$ integers from $\{1, 2, 3, \cdots, 2n\}$ (always) contains (at least) two consecutive integers.
- A bug can jump n units either to left or right in its n-th move for each positive integer n. That is the bug jumps either to left or right 1 unit in its first move, 2 units in its second move and so on. Can it get back to its starting positing after 2022 moves?
- Let $f(x) = 7x^3 - 5x + 17$. Show that $f(x)$ has no rational root. [Note that a real number α is a root of $f(x)$ if $f(\alpha) = 0$.]

Questions

- For any positive integer n, show that any selection $S \subseteq \{1, 2, 3 \cdots, 2n\}$ of $n+1$ integers contains (at least) integers a and b such that a divides b.
- For any positive integer n, there is an arrangement of integers $\{1, 2, 3, \cdots, n\}$ such that the arrangement contains no arithmetic progression of size 3.