# SNS COLLEGE OF TECHNOLOGY

Coimbatore-35
(An Autonomous Institution)
Accredited by NBA – AICTE and Accredited by NAAC – UGC with ‘A+’ Grade
Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

## DEPARTMENT OF MEHANICAL ENGINEERING

# 16ME401 FINITE ELEMENT ANALYSIS

IV YEAR VII SEM

## UNIT V ISOPARAMETRIC FORMULATION

*TOPIC* – Isoparametric elements-Example Problem 3

SNS *Design Thinkers*

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*Formula*

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta), \qquad N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta), \qquad N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$$

$$x = N_1 x_1 + N_2 x_2 + \boldsymbol{N_3}\, x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + \boldsymbol{N_3}\, y_3 + N_4 y_4$$

## Jacobian Matrix

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \varepsilon} & \dfrac{\partial y}{\partial \varepsilon} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \qquad [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{1}{4}\left[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4\right]$$

$$J_{22} = \frac{1}{4}\left[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4\right]$$

## Strain displacement Matrix:

$$[B]=\frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}\times\frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\eta) & 0 & (1-\varepsilon) \end{bmatrix}$$

Stress Strain Relationship Matrix:
For Plane stress condition:

$$[D]=\frac{E}{1-\gamma^2}\begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \frac{1-\gamma}{2} \end{bmatrix}$$

For Plane Strain condition:

$$[D]=\frac{E}{(1+\gamma)(1-2\gamma)}\begin{bmatrix} (1-\gamma) & \gamma & 0 \\ \gamma & (1-\gamma) & 0 \\ 0 & 0 & \frac{(1-2\gamma)}{2} \end{bmatrix}$$

E-Young's Modulus N/mm$^2$
$\gamma$ - Poisson's ratio

**Stiffness Matrix:**
**For Isoparametric Quadrilateral Element:**

$$[K]=t\iint[B]^T[D][B]\partial x\partial y$$

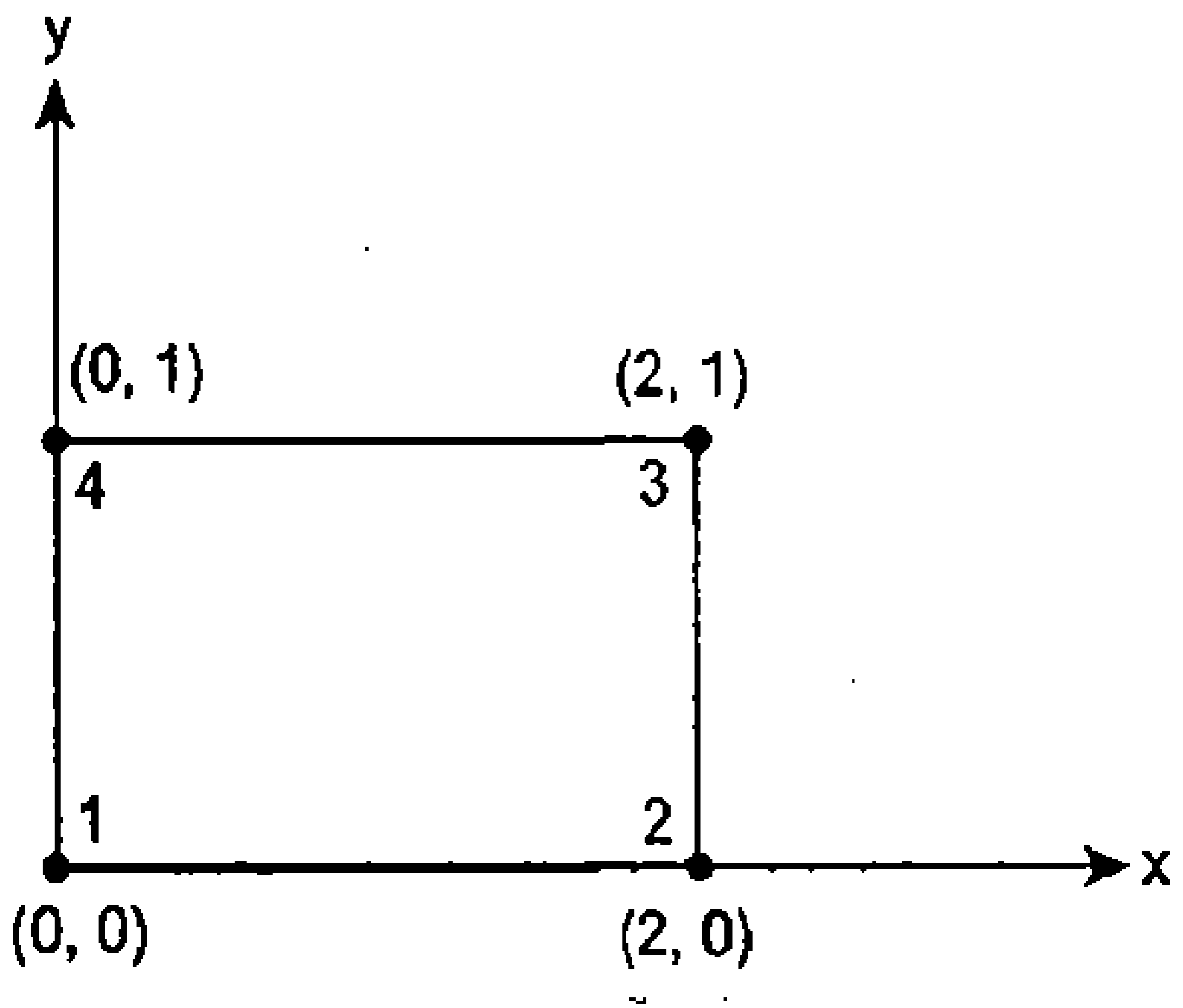
**For Natural Co-ordinates**

$$\int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]\times|J|\times\partial\varepsilon\times\partial\eta$$

$[B]$ ⟶ Strain Displacement matrix
$[D]$ ⟶ Stress – Strain Relationship matrix
$|J|$ ⟶ Determinant of the Jacobian matrix
ε,η ⟶ Natural Co – Ordinates
t ⟶ Thickness of the element - mm
$A$ ⟶ Area of the triangular element - mm$^2$

*A four noded rectangular element is shown in Fig. Determine the following:*

*1. Jacobian matrix;* *2. Strain-Displacement matrix;* *3. Element stresses.*




*Take* $E = 2 \times 10^5\ N/mm^2;\quad \nu = 0.25;\quad u = [0, 0, 0.003, 0.004, 0.006, 0.004, 0, 0]^T$

$\varepsilon = 0;\quad \eta = 0$

*Assume plane stress condition.*

Cartesian co-ordinates of point 1, 2, 3 and 4

$x_1 = 0; \quad y_1 = 0$

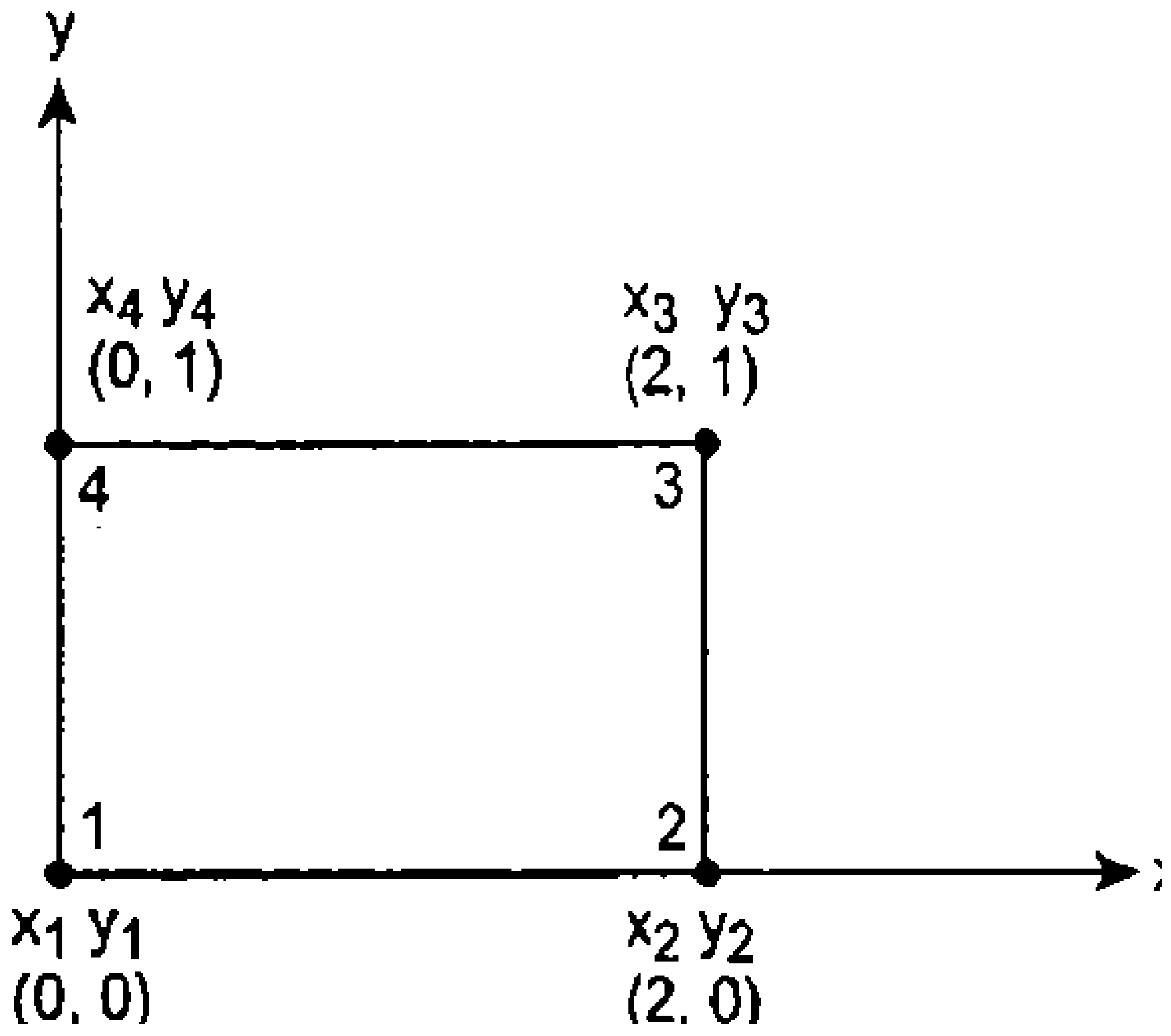
$x_2 = 2; \quad y_2 = 0$

$x_3 = 2; \quad y_3 = 1$

$x_4 = 0; \quad y_4 = 1$

Young's modulus, $E = 2 \times 10^5 \text{ N/m}^2$

Poisson's ratio, $\nu = 0.25$




Displacements, $u = \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$

Natural co-ordinates, $\varepsilon = 0, \ \eta = 0$

*To find:* 1. Jacobian matrix, J.

2. Strain-Displacement matrix, [B].

3. Element stress, σ.

**Jacobian matrix for quadrilateral element is given by,**

$$\Rightarrow [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

**Where,**

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{1}{4}\left[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4\right]$$

$$J_{22} = \frac{1}{4}\left[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4\right]$$

Cartesian co-ordinates of point

$x_1 = 0;$ $y_1 = 0$

$x_2 = 2;$ $y_2 = 0$

$x_3 = 2;$ $y_3 = 1$

$x_4 = 0;$ $y_4 = 1$

Natural co-ordinates, $\varepsilon = 0, \eta = 0$

*Substitute* $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, \varepsilon$ *and* $\eta$ *values in equation.*

$$\Rightarrow J_{11} = \frac{1}{4}[0+2+2-0]$$

$$J_{11} = 1$$

$$\Rightarrow J_{12} = \frac{1}{4}[0+0+1-1]$$

$$J_{12} = 0$$

$$\Rightarrow J_{21} = \frac{1}{4}[0-2+2+0]$$

$$J_{21} = 0$$

$$\Rightarrow J_{22} = \frac{1}{4}[-0-0+1+1]$$

$$J_{22} = 0.5$$

$$\Rightarrow [J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$\text{Jacobian matrix}, [J] = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\Rightarrow |J| = 1 \times 0.5 - 0$$

$$|J| = 0.5$$

**We know that, strain-Displacement matrix for quadrilateral element is,**

$$[B]=\frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}\times\frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

*Substitude* $J_{11}, J_{12}, J_{21}, J_{22}, |J|, \eta, \varepsilon$ *values,* $J_{11}=1 \quad J_{12}=0 \quad J_{21}=0 \quad J_{22}=0.5$

$\varepsilon=0, \eta=0$

$$[B]=\frac{1}{0.5}\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0.5 & 0 \end{bmatrix}\times\frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$=\frac{1}{0.5\times 4}\begin{bmatrix} -0.5 & 0 & 0.5 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & -0.5 & -1 & 0.5 & 1 & 0.5 & 1 & -0.5 \end{bmatrix}$$

$$=\frac{0.5}{0.5\times 4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

$$[B]=0.25\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}$$

We know that. $Element\ stress, \sigma = [D][B]\{u\}$

For plane stress condition,

Stress-strain relationship matrix,

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$[D] = \frac{2\times10^5}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$[D] = 213.33\times10^3\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$[D] = 213.33\times10^3\times0.25\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$[D] = 53.333\times10^3\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

## *Substitute $[D],[B]$, and $\{u\}$ values in equation*

$$\Rightarrow \{\sigma\} = 53.333\times10^{3}\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 0\\ 0 & 0 & 1.5\end{bmatrix}\times 0.25\begin{bmatrix}-1 & 0 & 1 & 0 & 1 & 0 & -1 & 0\\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2\\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1\end{bmatrix}\begin{Bmatrix}0\\0\\0.003\\0.004\\0.006\\0.004\\0\\0\end{Bmatrix}$$

$$\{\sigma\} = 53.333\times10^{3}\times 0.25\begin{bmatrix}-4 & -2 & 4 & -2 & 4 & 2 & -4 & 2\\ -1 & -8 & 1 & -8 & 1 & 8 & -1 & 8\\ -3 & -1.5 & -3 & 1.5 & 3 & 1.5 & 3 & -1.5\end{bmatrix}\begin{Bmatrix}0\\0\\0.003\\0.004\\0.006\\0.004\\0\\0\end{Bmatrix}$$

$$= 13.333\times10^{3}\begin{Bmatrix}0+0+(4\times0.003)+(-2\times0.004)+(4\times0.006)+(2\times0.004)+0+0\\ 0+0+(1\times0.003)+(-8\times0.004)+(1\times0.006)+(8\times0.004)+0+0\\ 0+0+(-3\times0.003)+(1.5\times0.004)+(3\times0.006)+(1.5\times0.004)+0+0\end{Bmatrix}$$

$$\{\sigma\} = 13.333\times10^{3}\begin{Bmatrix}0.036\\0.009\\0.021\end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix}480\\120\\280\end{Bmatrix} N/m^{2}$$

**Result**

1. $Jacobian\ matrix,\ [J] = \begin{bmatrix}1 & 0\\0 & 0.5\end{bmatrix}$

2. $strain - Displacement\ matrix,\ [B] = 0.25\begin{bmatrix}-1 & 0 & 1 & 0 & 1 & 0 & -1 & 0\\0 & 2 & 0 & -2 & 0 & 2 & 0 & 2\\-2 & -1 & -2 & 1 & 2 & 1 & 2 & 1\end{bmatrix}$

3. $Stress,\ \{\sigma\} = \begin{Bmatrix}480\\120\\280\end{Bmatrix} N/m^{2}$

# *Thank you*