Natural co-ordinate systems

Session objectives

At the end of this session, the learner will be able to

1) understand 2D **ISOPARAMETRIC** problems in engineering
2) understand and analyze Natural co-ordinate systemsproblems

Learning Outcome

Students should be able to

1) understand 2D **ISOPARAMETRIC** problems in engineering
2) understand and analyze Natural co-ordinate systemsproblems

Teaching learning material

- Board/White Board and Markers
- Presentation/PPT

Natural co-ordinate systems

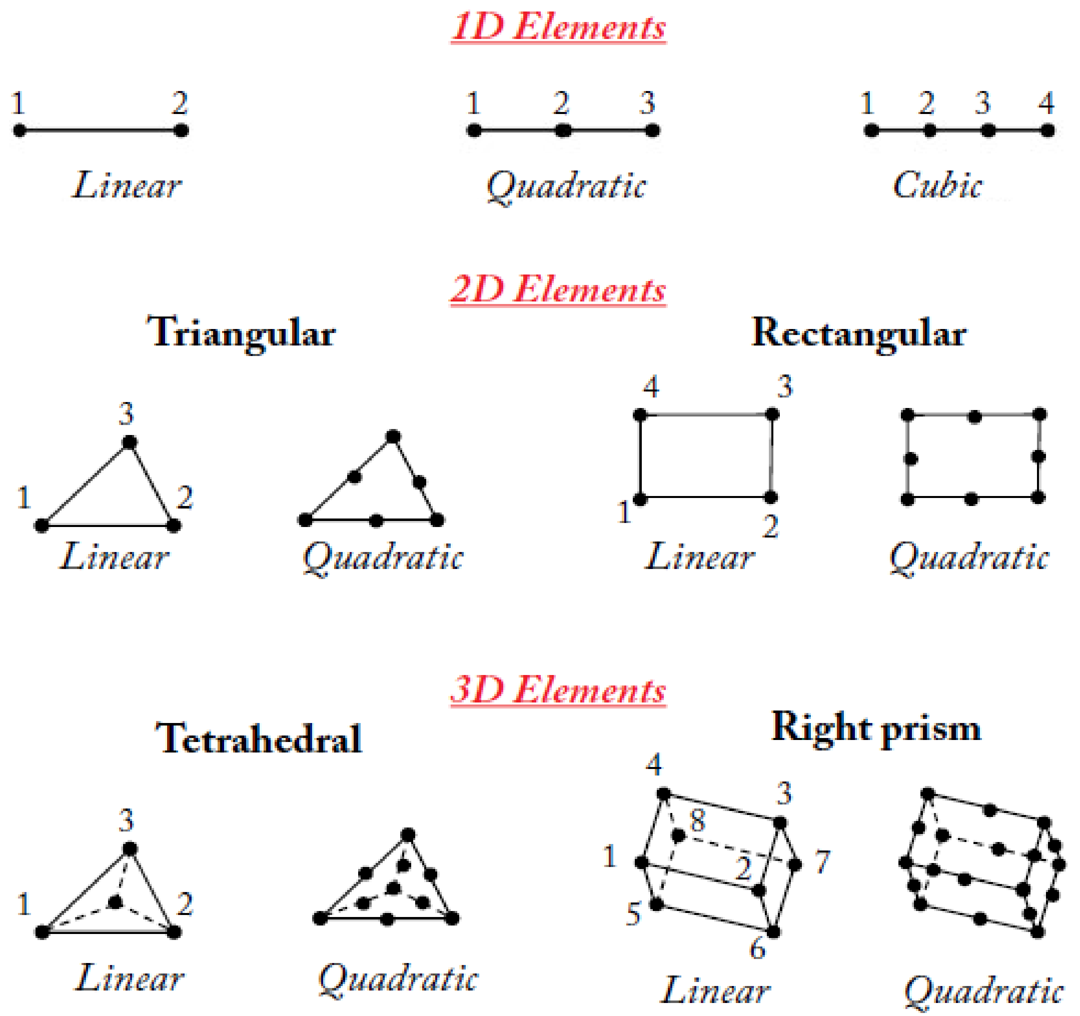


Figure : Higher-order elements in 1D, 2D, and 3D.

Natural co-ordinate systems

In FEM, different coordinate systems are used as a reference in order to specify different quantities easily and meaningfully. In the global coordinate system, the node location, element orientation, the loads and the boundary conditions are specified conveniently. Also, the solution (nodal variables) is generally represented in the global coordinate system.

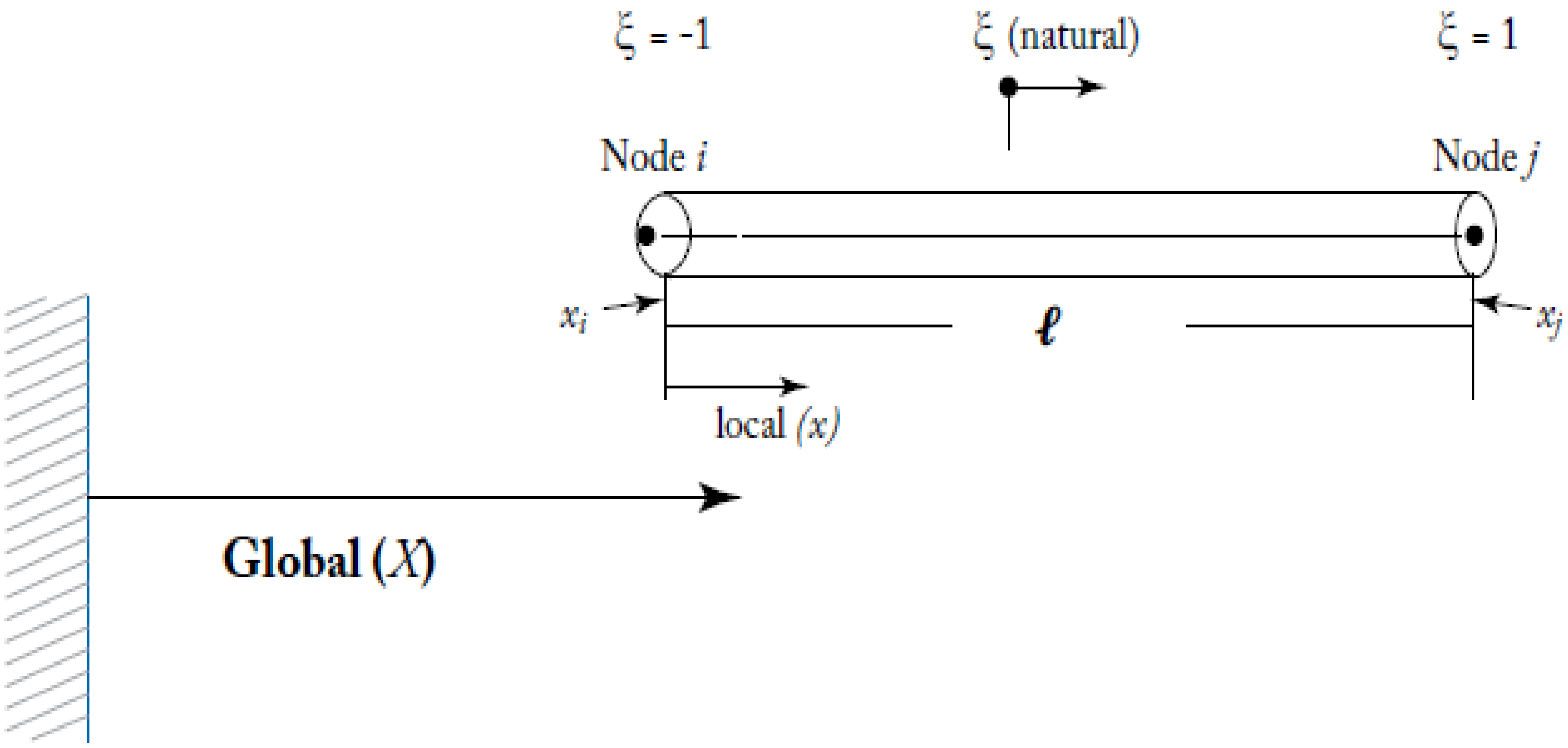
A local coordinate system is very convenient to attach to an element in order to simplify the derivation of element stiffness matrix through algebraic manipulations.

Natural co-ordinate systems

A natural coordinate system is a local coordinate system that uses a dimensionless parameter whose limits are within -1 to + 1. therefore, natural coordinates are dimensionless and are usually represented with respect to the element rather than the global coordinate system. The use of a natural coordinate system simplifies the integration when evaluating a stiffness matrix using numerical integration formulas. Certain types of finite elements known as isoparametric elements use natural coordinates and play a crucial role in modeling curved boundaries.

In general, global, local, and natural coordinate systems are related through transformation. Consider a 1D system with a global coordinate system (X) local coordinate system (*x*) and a natural coordinate system ( ) as shown in Fig.

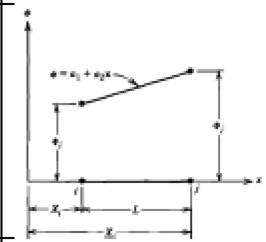
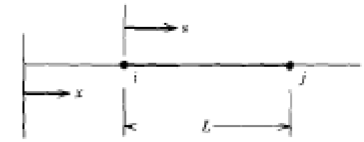
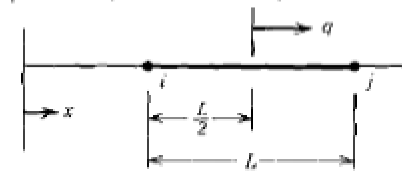
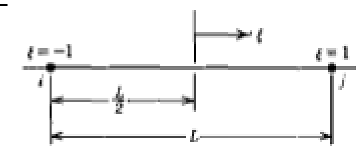
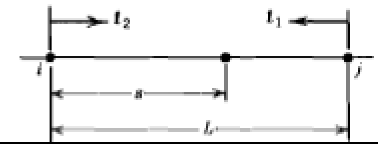
Natural co-ordinate systems



Different coordinate systems—global (X), local (x), and natural (x).

Natural co-ordinate systems

Coordinate Systems and Limits of Integration for the One-Dimensional Element

Type of System	Coordinate Variable	Shape Functions	Limits of Integration	coordinate systems for the one-dimensional element.
Global	x	$N_i = \frac{X_j - x}{L}$, $N_j = \frac{x - X_i}{L}$	X_i, X_j	
Local	s	$N_i = 1 - \frac{s}{L}$, $N_j = \frac{s}{L}$	0, L	
Local	q	$N_i = \left(\frac{1}{2} - \frac{q}{L}\right)$, $N_j = \left(\frac{1}{2} + \frac{q}{L}\right)$	$-\frac{L}{2}$, $\frac{L}{2}$	
Natural	ξ	$N_i = \frac{1}{2}(1 - \xi)$, $N_j = \frac{1}{2}(1 + \xi)$	−1, 1	
Natural	ℓ_2	$N_i = \ell_1$, $N_j = \ell_2$	0, 1	

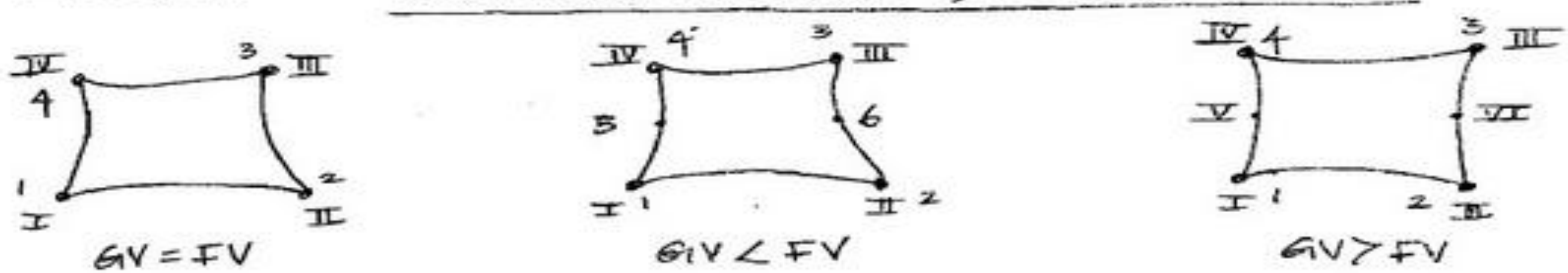
Natural co-ordinate systems

Isoparametric element: The element whose geometry and field variables are described by same interpolation function of same order is known as Isoparametric element.

(ie) Geometric variable = Field variable

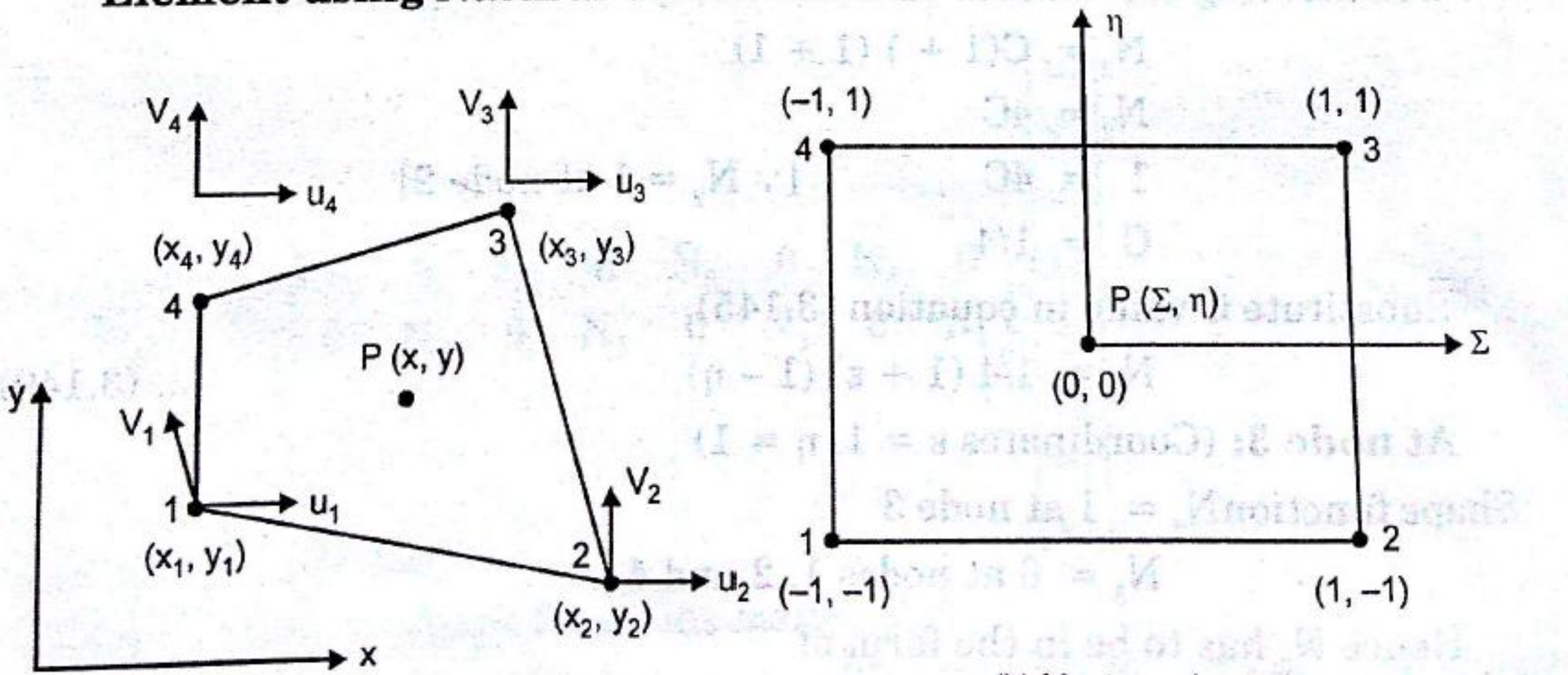
- **Subparametric element:** The element whose geometry is described by lower order interpolation model compared to field variable. Geometric variable < Field variable

Superparametric element: In these elements Geometry is described by higher order interpolation model than the field variable. Geometric variable > Field variable



Natural co-ordinate systems

Derivation of Shape Functions for a Four Noded Quadrilateral Element using Natural Coordinates



(a) General quadrilateral element

(b) Master - element
(i.e,. Isoparametric quadrilaterals element)

Natural co-ordinate systems

Consider a four noded quadrilateral element specified by global coordinate system as shown in fig.

The master (or) parent element is defined in ε and η coordinates i.e., natural coordinates. ε is varying from – 1 to 1 and η is also varying – 1 to 1.

To determine the shape functions for master element, we can adopt the concept of serendipity approach

Natural co-ordinate systems

We know that, shape function value is unity at its own node and its value is zero at other nodes.

At node 1: (Co -ordinates $\varepsilon = -1, \eta = -1$)

Shape function $N_1 = 1$ at node 1 and

$N_1 = 0$ at node 2, 3 and 4

Hence N_1, has to be in the form of

$$N_1 = C(1-\varepsilon)(1-\eta)$$

where, C is constant

Substituting coordinates values $\varepsilon = -1$ and $\eta = -1$ in equation (3.146), we get

$$N_1 = C(1+1)(1+1)$$

$$N_1 = 4C$$

$$1 = 4C \qquad [\because N_1 = 1 \text{ at node } 1]$$

$$C = 1/4$$

Substitute C value in equation (3.146)

$$N = 1/4\,(1-\varepsilon)(1-\eta)$$

Natural co-ordinate systems

At node 2: (coordinates $\varepsilon = 1, \eta = -1$)

Shape function $N_2 = 1$ at node 2

$N_2 = 0$ at nodes 1, 3 and 4

Hence N_2 has to be in the form of

$$N_2 = C(1+\varepsilon)(1-\eta)$$

Substituting coordinates values $\varepsilon = 1, \eta = -1$ in equation

$$N_2 = C(1+1)(1+1)$$

$$N_2 = 4C$$

$$1 = 4C \qquad [\because N_2 = 1 \text{ at node } 2]$$

$$C = 1/4$$

Substitute C value in equation (3.145)

$$N_2 = 1/4\,(1+\varepsilon)(1-\eta)$$

Natural co-ordinate systems

At node 3: (Coordinates $\varepsilon = 1, \eta = 1$)

Shape function $N_3 = 1$ at node 3

$N_3 = 0$ at nodes 1, 2 and 4

Hence N_3 has to be in the form of

$$N_3 = C(1 + \varepsilon)(1 + \eta)$$

Substituting coordinates values $\varepsilon = 1, \eta = 1$ in equation

$$N_3 = C(1 + 1)(1 + 1)$$

$$N_3 = 4C$$

$$1 = 4C \qquad [\because N_3 = 1 \text{ at node } 3]$$

$$C = 1/4$$

Substitute C value in equation (3.150)

$$N_3 = 1/4\,(1 + \varepsilon)(1 + \eta)$$

Natural co-ordinate systems

At node 4: (coordinates $\varepsilon = -1, \eta = 1$)

Shape function $N_4 = 1$ at node 4

$N_4 = 0$ at nodes 1, 2 and 3

Hence N_4 has to be in the form of

$$N_4 = C(1-\varepsilon)(1+\eta)$$

Substituting coordinates values $\varepsilon = -1, \eta = 1$ in equation

$$N_4 = C(1+1)(1+1)$$

$$N_4 = 4C$$

$$1 = 4C \qquad [\because N_4 = 1 \text{ at node } 4]$$

$$C = 1/4$$

Substitute C value in equation ( )

$$N_4 = 1/4(1-\varepsilon)(1+\eta)$$

Natural co-ordinate systems

∴ The displacement u can be expressed as

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$

Similarly displacement v can be expressed as

$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$$

we get displacement at any point 'ρ' inside the element as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\{u(\varepsilon, \eta)\} = [N]\{u\}$$

Where, $[N]$ – shape functions matrix and

N – $f(\varepsilon, \eta)$[Refer equation

$\{u\}$ – Nodal displacement vector

Natural co-ordinate systems

In the isoparametric formulation, we can also use the same shape functions N_1 to N_4 to express the coordinates of the point P within the element in terms of nodal coordinates specified by global system.

i.e., $x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$

$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$ ... ( )

where N_1, N_2, N_3 and N_4 are described in equations ( ), ( ), ( ) and ( ).

Note 1:

The method of locating or identifying the point P in the actual quadrilateral element fig. (5.11 a) by global coordinate system is referred as **"Mapping" of the element**, i.e, the point P(ε, η) has been mapped into P(x, y) by using the equation ( ).

Natural co-ordinate systems

Note 2:

We can check the properties of shape functions such that,

$$N_1 + N_2 + N_3 + N_4 = 1$$

i.e., $N_1 + N_2 + N_3 + N_4 = 1/4[1 - \varepsilon - \eta + \varepsilon\eta + 1 + \varepsilon - \eta - \varepsilon\eta + 1 + \varepsilon + \eta + \varepsilon\eta + 1 - \varepsilon + \eta - \varepsilon\eta]$

$$= 4/4$$

$$= 1 \quad \text{(First condition is satisfied)}$$

Now, at node 1, $\varepsilon = -1, \eta = -1$

$$N_1 = 1/4\,(1 + 1)\,(1 + 1)$$

$$= 1$$

At node 2, $\varepsilon = 1, \eta = -1$

$$\therefore \quad N_1 = 1/4\,(1 - 1)\,(1 + 1)$$

$$N_1 = 0$$

Similarly N_1 for node 3 and node 4 are equal to zero. Hence the second condition is also satisfied.

Natural co-ordinate systems

Thank You