Type 1:

RHS $= e^{ax}$

Replace D by a.

Q. Solve $(D^2+1)y = e^{-x}$

Soln.

The Auxiliary eqn. is $m^2+1 = 0$

$m^2 = -1$

$m = \pm i$

$\therefore$ The roots are imaginary.

$CF = e^{0x}[A\cos x + B\sin x]$

$CF = A\cos x + B\sin x$

$PI = \frac{1}{D^2+1} e^{-x}$

$= \frac{1}{(-1)^2+1} e^{-x}$ Replace $D \to a = -1$

$= \frac{1}{2} e^{-x}$

$PI = \frac{e^{-x}}{2}$

$\therefore$ The soln. is $y = CF + PI$

$y = A\cos x + B\sin x + \frac{e^{-x}}{2}$

2]. Solve $(D^2+4D+4)y = 11e^{-2x}$

Soln.

The auxiliary eqn. is, $m^2+4m+4=0$

$(m+2)^2 = 0$

$m = -2, -2$

The roots are real and same.

$CF = (A+Bx)e^{-2x}$

$PI = \frac{1}{D^2+4D+4} 11e^{-2x}$

Replace $D \to a = -2$

$= 11 \frac{1}{(-2)^2+4(-2)+4} e^{-2x}$

$= 11 \frac{1}{4-8+4} e^{-2x}$

$= 11x \frac{1}{2D+4} e^{-2x}$

[If Dr. → 0, then multiply x in the Nr. and differentiate the Dr. w.r. to D]

$= 11x \frac{1}{2(-2)+4} e^{-2x}$

$= 11x^2 \frac{1}{2} e^{-2x}$

$= \frac{11x^2}{2} e^{-2x}$

∴ The general soln. is,

$y = CF + PI$

$= (A+Bx)e^{-2x} + \frac{11x^2}{2}e^{-2x}$

3]. Solve $(D^2-2D+1)y = \cosh x$

Soln.

Given $(D^2-2D+1)y = \frac{e^x+e^{-x}}{2}$

$= \frac{1}{2}[e^x+e^{-x}]$

$= \frac{e^x}{2} + \frac{e^{-x}}{2}$

AE

$$m^2-2m+1=0$$
$$m=1,1$$

CF

$$CF=(A+Bx)e^x$$

$$PI_1=\frac{1}{D^2-2D+1}\frac{e^x}{2}$$

$$=\frac{1}{2}\frac{1}{1^2-2(1)+1}e^x \qquad D\to 1$$

$$=\frac{x}{2}\frac{1}{2D-2}e^x$$

$$=\frac{x^2}{2}\cdot\frac{1}{2}e^x$$

$$PI_1=\frac{x^2}{4}e^x$$

$$PI_2=\frac{1}{D^2-2D+1}\frac{e^{-x}}{2}$$

$$=\frac{1}{2}\frac{1}{(-1)^2-2(-1)+1}e^{-x}$$

$$=\frac{1}{2}\frac{1}{1+2+1}e^{-x}$$

$$PI_2=\frac{1}{8}e^{-x}$$

The general soln. is

$$y=CF+PI_1+PI_2$$
$$y=(A+Bx)e^x+\frac{x^2}{4}e^x+\frac{1}{8}e^{-x}$$

Type 2:

$RHS = \sin(ax+b)$

or

$\cos(ax+b)$

Replace $D^2 \rightarrow -a^2$

1. Solve $(D^2+3D+2)y = \sin 3x$

Soln.

CF

$$m^2+3m+2 = 0$$

$$(m+1)(m+2) = 0$$

$$m = 1, 2$$

$$\therefore CF = Ae^{x} + Be^{2x}$$

$$PI = \frac{1}{D^2-3D+2} \sin 3x \qquad D^2 = -a^2 = -9$$

$$= \frac{1}{-9-3D+2} \sin 3x$$

$$= \frac{1}{-3D-7} \sin 3x$$

$$= \frac{1}{(-3D-7)} \frac{(-3D+7)}{(-3D+7)} \sin 3x$$

$$= \frac{-3D+7}{9D^2-49} \sin 3x$$

$$= \frac{-3D+7}{9(-9)-49} \sin 3x \qquad D^2 \rightarrow -9$$

$$= \frac{-3D+7}{-130} \sin 3x.$$

$$= \frac{-1}{130} [(-3D+7) \sin 3x]$$

$$= \frac{-1}{130} [-3D \sin 3x + 7 \sin 3x]$$

$$= \frac{-1}{130} [-3(3) \cos 3x + 7 \sin 3x]$$

$$= \frac{1}{1^2-3(1)+2}\, 2e^{x} \qquad D \to a = 1$$

$$= x\frac{1}{2D-3}\, 2e^{x}$$

$$= x\frac{1}{2(1)-3}\, 2e^{x}$$

$$= \frac{x}{-1}\, 2e^{x}$$

$$= -2xe^{x}$$

$\therefore$ the soln. is,

$$y = CF + PI_1 + PI_2$$

$$= Ae^{x} + Be^{2x} - \frac{1}{20}\left[6\sin(2x+3) + 2\cos(2x+3)\right] - 2xe^{x}$$

3). Find the PI of $(D^2+5D+6)y = \sin 3x \cos x$

Soln.

Given that $(D^2+5D+6)\,y = \sin 3x \cos x$

$$= \frac{1}{2}\left[\sin 4x + \sin 2x\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$PI_1 = \frac{1}{D^2+5D+6}\, \frac{1}{2}\sin 4x$$

$$= \frac{1}{-16+5D+6}\, \frac{1}{2}\sin 4x \qquad D^2 \to -a^2 = -4^2 = -16$$

$$= \frac{1}{5D-10}\, \frac{1}{2}\sin 4x$$

$$= \frac{1}{2}\, \frac{5D+10}{25D^2-100}\sin 4x$$

$$= \frac{1}{2}\, \frac{5D\sin 4x + 10\sin 4x}{25(-16)-100}$$

$$= \frac{1}{2\times(-500)}\left[20\cos 4x + 10\sin 4x\right]$$

$$PI_1 = \frac{-1}{+100}\left[2\cos 4x + \sin 4x\right]$$

$$PI_2 = \frac{1}{D^2+5D+6}\ \frac{1}{2}\sin 2x$$

$$= \frac{1}{2}\ \frac{1}{-4+5D+6}\sin 2x \qquad D^2 \to -a^2 = -4$$

$$= \frac{1}{2}\ \frac{5D-2}{25D^2-4}\sin 2x$$

$$= \frac{1}{2}\ \frac{\left[5D\sin 2x - 2\sin 2x\right]}{25(-4)-4}$$

$$PI_2 = -\frac{1}{104}\left[5\cos 2x - \sin 2x\right]$$

The soln. is,

$$y = \cancel{CF+}\ PI_1 + PI_2$$

$$= \frac{-1}{100}\left[2\cos 4x + \sin 4x\right] - \frac{1}{104}\left[5\cos 2x - \sin 2x\right]$$

Type 3: RHS $= x^n$

1). $(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$

2). $(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$

3). $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$

4). $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$

Q. Solve $(D^2+2)y = x^2$

Soln.

AE $\quad m^2 + 2 = 0$

$m^2 = -2$

$m = \pm\sqrt{2}\, i$

$\alpha \pm i\beta \Rightarrow \alpha = 0, \ \beta = \sqrt{2}$

$CF = A\cos\sqrt{2}x + B\sin\sqrt{2}x$

$$PI = \frac{1}{D^2+2} x^2$$

$$= \frac{1}{2\left[1+\frac{D^2}{2}\right]} x^2$$

$$= \frac{1}{2}\left[1+\frac{D^2}{2}\right]^{-1} x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+\frac{D^4}{4} - \cdots\right] x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}\right] x^2$$

$D^2 x^2 = 2x \to 2$

$$= \frac{1}{2}\left[x^2 - \frac{D^2 x^2}{2}\right] = \frac{1}{2}\left[x^2 - \frac{2}{2}\right]$$

$$= \frac{1}{2}\left[x^2 - 1\right]$$

$\therefore$ The soln. is, $y = CF + PI$

$$= A\cos\sqrt{2}x + B\sin\sqrt{2}x + \frac{1}{2}\left[x^2 - 1\right]$$

2]. Solve $(D^2+3D+2)y = x^2$

Soln.

AE $m^2+3m+2=0$

$(m+1)(m+2)=0$

$m=-1,-2$

$CF = Ae^{x}+Be^{-2x}$

$PI = \frac{1}{D^2+3D+2}x^2$

$= \frac{1}{2\left[1+\frac{D^2+3D}{2}\right]}x^2$

$= \frac{1}{2}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1}x^2$

$= \frac{1}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2\right]x^2$

$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{9D^2}{4}\right]x^2$

$= \frac{1}{2}\left[x^2-\frac{D^2x^2}{2}-\frac{3D}{2}x^2+\frac{9D^2}{4}x^2\right]$

x^2
$D' = 2x$
$D'' = 2$

$= \frac{1}{2}\left[x^2-\frac{2}{2}-\frac{3(2x)}{2}+\frac{9(2)}{4}\right]$

$= \frac{1}{2}\left[x^2-1-3x+9/2\right]$

$PI = \frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]$

The Soln. is, $y = CF+PI$

$y = Ae^{x}+Be^{-2x}+\frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]$

Type-4

$RHS = e^{ax}\phi(x)$ where $\phi(x) = \sin bx$ or $\cos bx$ or x^m

Replace $D \to D+a$

1. Solve $(D^2-4D+3)y = e^{x}\cos 2x$

Soln.

$m^2-4m+3=0$

$m=1,3$

$CF = Ae^{x}+Be^{3x}$

$PI = \dfrac{1}{D^2-4D+3}e^{x}\cos 2x$

$= e^{x}\dfrac{1}{(D+1)^2-4(D+1)+3}\cos 2x \quad D \to D+a = D+1$

$= e^{x}\dfrac{1}{D^2+1+2D-4D-4+3}\cos 2x$

$= e^{x}\dfrac{1}{D^2-2D}\cos 2x$

$= e^{x}\dfrac{1}{-4-2D}\cos 2x \quad D^2 \to -a^2 = -2^2 = -4$

$= e^{x}\dfrac{-2D+4}{4D^2-16}\cos 2x$

$= e^{x}\dfrac{-2D\cos 2x+4\cos 2x}{4(-4)-16}$

$= \dfrac{e^{x}}{-32}[4\sin 2x+4\cos 2x]$

$= -\dfrac{e^{x}}{8}[\sin 2x+\cos 2x]$

$\therefore$ The Soln. is $y = CF + PI$

$y = Ae^{x}+Be^{3x}-\dfrac{e^{x}}{8}[\sin 2x+\cos 2x]$

2]. Find the PI of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = xe^{-2x}$

Soln.

Given that $(D^2+4D+4)y = xe^{-2x}$

$$PI = \frac{1}{D^2+4D+4} e^{-2x} x$$

$$= e^{-2x} \frac{1}{(D-2)^2+4(D-2)+4} x \qquad D \to D+a = D-2$$

$$= e^{-2x} \frac{1}{D^2+4-4D+4D-8+4} x$$

$$= e^{-2x} \frac{1}{D^2} x$$

$$PI = \frac{e^{-2x} x^3}{6}$$

$$\frac{1}{D}x = \int x = \frac{x^2}{2}$$

$$\frac{1}{D^2} = \frac{x^3}{6}$$

HW 1]. Solve $(D^2-4D-5)y = xe^{x}$

2]. Solve $(D^2+4D+4)y = e^{-2x} x^2$

3]. $(D^2+4D+4)y = e^{-2x}\sin x$

Type - 5

Case 1: $RHS = x\,\phi(x)$ where $\phi(x) = \sin ax$ or $\cos ax$

$$PI = x\frac{1}{f(D)}\phi(x) - \frac{f'(D)}{[f(D)]^2}\phi(x).$$

Case 2:

$RHS = x^n \phi(x)$

i). $PI =$ Imaginary part $\left[\frac{1}{f(D)} x^n e^{iax}\right]$ if $\phi(x) = \sin ax$

ii). $PI =$ Real part $\frac{1}{f(D)} x^n e^{iax}$ if $\phi(x) = \cos ax$

1). Solve $(D^2+4)y = x \sin x$

Soln.

$m^2+4=0$

$m^2 = -4$

$m = \pm 2i$

$\alpha = 0, \quad \beta = 2$

$CF = A\cos 2x + B\sin 2x$

$PI = \frac{1}{D^2+4} x \sin x$

$= x \frac{1}{D^2+4} \sin x - \frac{2D}{(D^2+4)^2} \sin x .$

$= x \frac{1}{-1+4} \sin x - \frac{4\cos x}{(-1+4)^2}$ $\quad D^2 \to -a^2 = -1^2 = -1$

$= \frac{x \sin x}{3} - \frac{4\cos x}{9}$

$\therefore$ The soln. is $y = CF + PI$

$y = A\cos 2x + B\sin 2x + \frac{x\sin x}{3} - \frac{4\cos x}{9}$

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2). Solve $(D^2-2D+1)y = xe^x \sin x$

Soln.

$m^2-2m+1=0$

$m = 1, 1$

$CF = (A+Bx)e^x$

$PI = \frac{1}{D^2-2D+1} e^x x \sin x$

$= e^x \frac{1}{(D+1)^2-2(D+1)+1} x \sin x$ $\quad D \to D+a = D+1$

$= e^x \frac{1}{D^2+1+2D-2D-2+1} x \sin x$

$$= e^{x} \frac{1}{D^2} x \sin x$$

$$= e^{x}\left[x \frac{1}{D^2} \sin x - \frac{2D}{D^4} \sin x\right]$$

$$= e^{x}\left[x \frac{1}{-1} \sin x - \frac{2\cos x}{(-1)^2}\right]$$

$$PI = -x e^{x} \sin x - 2 e^{x} \cos x$$

The soln. is,

$$y = CF + PI$$

$$= (A + Bx) e^{x} - x e^{x} \sin x - 2 e^{x} \cos x$$