SNS COLLEGE OF TECHNOLOGY

Coimbatore-26
An Autonomous Institution

Accredited by NBA – AICTE and Accredited by NAAC – UGC with 'A++' Grade
Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

19ECT212 – CONTROL SYSTEMS

II YEAR/ IV SEMESTER

UNIT V – STATE VARIABLE ANALYSIS

TOPIC 3,4 TRANSFER FUNCTION FROM STATE VARIABLE REPRESENTATION & SOLUTIONS OF THE STATE EQUATIONS

OUTLINE

- REVIEW ABOUT PREVIOUS CLASS
- INTRODUCTION
- BASIC CONCEPTS OF STATE SPACE MODEL
- STATE SPACE MODEL FROM **DIFFERENTIAL EQUATION**
- STATE SPACE MODEL FROM **TRANSFER FUNCTION & EXAMPLE**
- ACTIVITY
- TRANSFER FUNCTION **FROM STATE SPACE MODEL-EXAMPLE**.
- STATE TRANSITION MATRIX & PROPERTIES
- SUMMARY

INTRODUCTION

The **state space model** of Linear Time-Invariant (LTI) system can be represented as,

$$\dot{X}=AX+BU$$

$$Y=CX+DU$$

The first and the second equations are known as state equation and output equation respectively.

Where,

- X and $\dot{X}$ are the state vector and the differential state vector respectively.
- U and Y are input vector and output vector respectively.
- A is the system matrix.
- B and C are the input and the output matrices.
- D is the feed-forward matrix.

BASIC CONCEPTS OF STATE SPACE MODEL

BASIC TERMINOLOGY:

State

It is a group of variables, which summarizes the history of the system in order to predict the future values (outputs).

State Variable

The number of the state variables required is equal to the number of the storage elements present in the system.

Examples – current flowing through inductor, voltage across capacitor

State Vector

It is a vector, which contains the state variables as elements.

two mathematical models of the control systems.

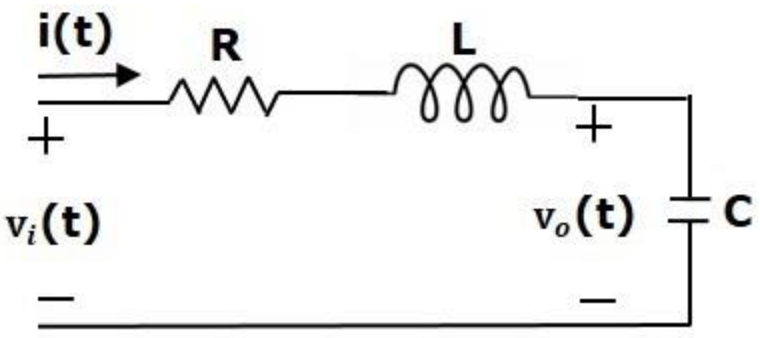
1.differential equation model
2.transfer function model.

The state space model can be obtained from any one of these two mathematical models.

STATE SPACE MODEL FROM DIFFERENTIAL EQUATION

Consider the following series of the RLC circuit.

It is having an input voltage, vi(t) and the current flowing through the circuit is i(t).



There are two storage elements (inductor and capacitor) in this circuit.

So, the number of the state variables is equal to two and these state variables are the current flowing through the inductor, i(t) and the voltage across capacitor, vc(t).

STATE SPACE MODEL FROM DIFFERENTIAL EQUATION…

From the circuit, the output voltage, v0(t) is equal to the voltage across capacitor, vc(t).

$$v0(t) = v\,c(t)$$

Apply KVL around the loop.

$$vi(t)=R\,i(t)+L\,di(t)/dt + vc(t)$$

$$\Rightarrow d\,i(t)/\,dt \;= -R\,i(t)\,/\,L \;-\; vc(t)\,/\,L \;+ vi(t)/\,L$$

The voltage across the capacitor is –

$$vc(t)=1/C\int i(t)dt$$

Differentiate the above equation with respect to time.

$$d\,vc(t)/dt = \; i(t)C$$

STATE SPACE MODEL FROM DIFF. EQN…

State vector, $X = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$

Differential state vector, $\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix}$

We can arrange the differential equations and output equation into the standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_i(t)]$$

$$Y = [0 \quad 1] \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, C = [0 \quad 1] \; and \; D = [0]$$

STATE SPACE MODEL FROM TRANSFER FUNCTION

Consider the two types of transfer functions based on the type of terms present in the numerator.

1. Transfer function having constant term in Numerator.

1. Transfer function having polynomial function of ‘s’ in Numerator.

STATE SPACE MODEL FROM TRANSFER FUNCTION

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

Rearrange, the above equation as

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0)Y(s) = b_0 U(s)$$

Apply inverse Laplace transform on both sides.

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_0 u(t)$$

Let

$$y(t) = x_1$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = x_3 = \dot{x}_2$$

STATE SPACE MODEL FROM TRANSFER FUNCTION

$$\frac{d^{n-1}y(t)}{dt^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{d^n y(t)}{dt^n} = \dot{x}_n$$

and $u(t) = u$

Then,

$$\dot{x}_n + a_{n-1}x_n + \ldots + a_1x_2 + a_0x_1 = b_0u$$

From the above equation, we can write the following state equation.

$$\dot{x}_n = -a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + b_0u$$

The output equation is -

$$y(t) = y = x_1$$

STATE SPACE MODEL FROM TRANSFER FUNCTION

The state space model is -

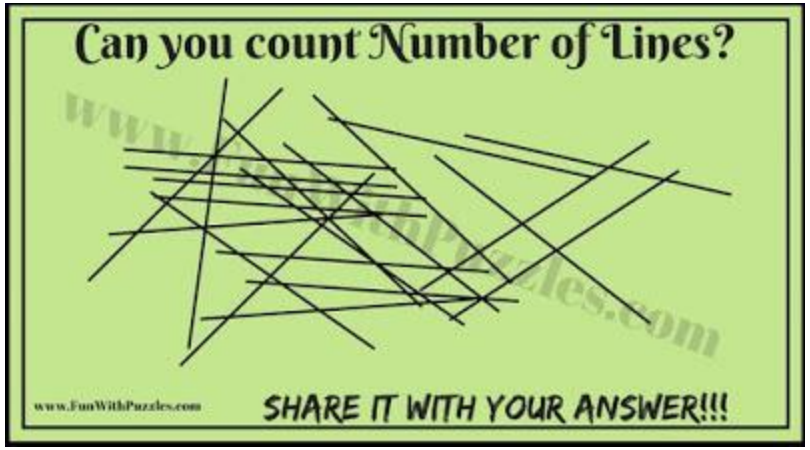
$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

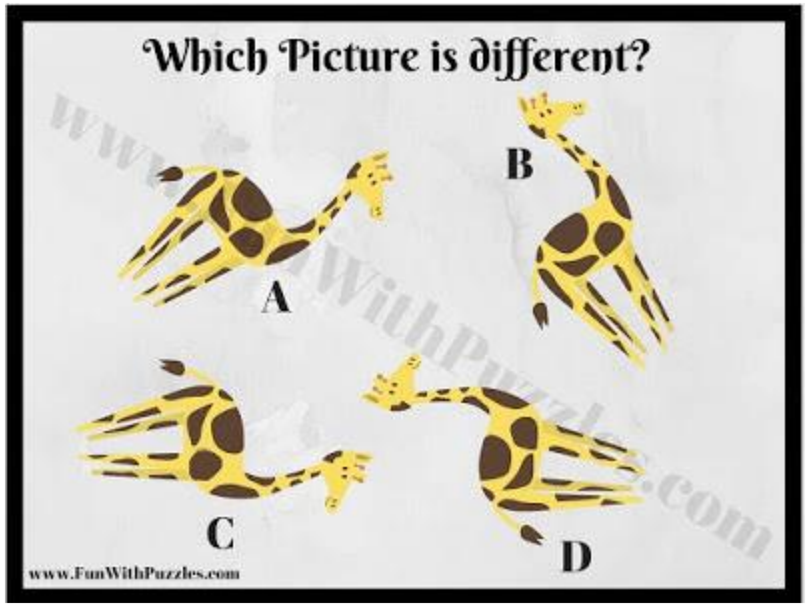
$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

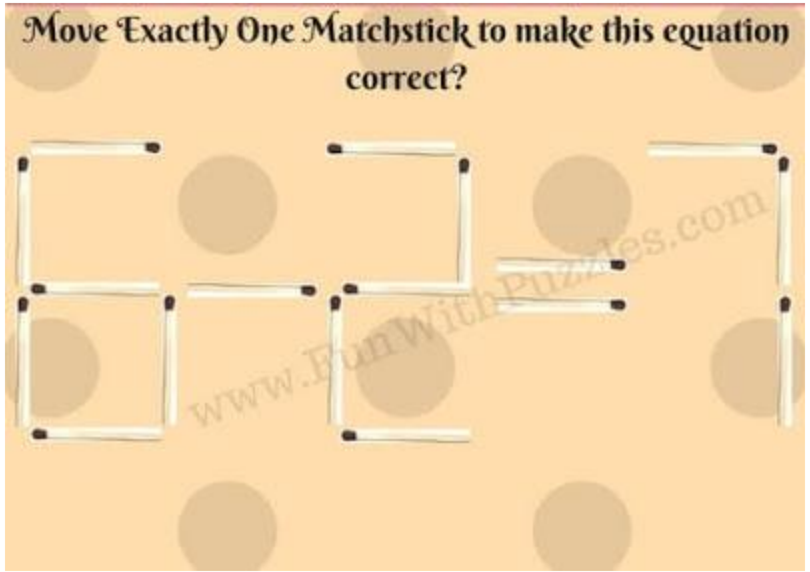
$$Y = [1 \quad 0 \quad \dots \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Here, $D = [0]$.

ACTIVITY


Can you count Number of Lines?
www.FunWithPuzzles.com
SHARE IT WITH YOUR ANSWER!!!


Which Picture is different?
A
B
C
D
www.FunWithPuzzles.com


Move Exactly One Matchstick to make this equation correct?


CAN YOU SOLVE THIS REBUS?
ASLEEP
SHARE IF YOU SOLVED IT!!!

STATE SPACE MODEL FROM TRANSFER FUNCTION-EX…

Example

Find the state space model for the system having transfer function.

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

Rearrange, the above equation as,

$$(s^2 + s + 1)Y(s) = U(s)$$

Apply inverse Laplace transform on both the sides.

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = u(t)$$

Let

$$y(t) = x_1$$

$$\frac{dy(t)}{dt} = x_2 = \dot{x}_1$$

and $u(t) = u$

Then, the state equation is

$$\dot{x}_2 = -x_1 - x_2 + u$$

STATE SPACE MODEL FROM TRANSFER FUNCTION-EX...

The output equation is

$$y(t) = y = x_1$$

The state space model is

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

STATE SPACE MODEL FROM TRANSFER FUNCTION

Transfer function having polynomial function of 's' in Numerator

Consider the following transfer function of a system

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \left(\frac{1}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}\right)(b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0)$$

The above equation is in the form of product of transfer functions of two blocks, which are cascaded.

$$\frac{Y(s)}{U(s)} = \left(\frac{V(s)}{U(s)}\right)\left(\frac{Y(s)}{V(s)}\right)$$

Here,

$$\frac{V(s)}{U(s)} = \frac{1}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

Rearrange, the above equation as

$$(s^n + a_{n-1} s^{n-1} + \ldots + a_0) V(s) = U(s)$$

STATE SPACE MODEL FROM TRANSFER FUNCTION

Apply inverse Laplace transform on both the sides.

$$\frac{d^n v(t)}{dt^n} + a_{n-1}\frac{d^{n-1} v(t)}{dt^{n-1}} + \ldots + a_1\frac{dv(t)}{dt} + a_0 v(t) = u(t)$$

Let

$$v(t) = x_1$$

$$\frac{dv((t)}{dt} = x_2 = \dot{x}_1$$

$$\frac{d^2 v(t)}{dt^2} = x_3 = \dot{x}_2$$

STATE SPACE MODEL FROM TRANSFER FUNCTION

$$\frac{d^{n-1}v(t)}{dt^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{d^n v(t)}{dt^n} = \dot{x}_n$$

and $u(t) = u$

Then, the state equation is

$$\dot{x}_n = -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + u$$

Consider,

$$\frac{Y(s)}{V(s)} = b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0$$

Rearrange, the above equation as

$$Y(s) = (b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0) V(s)$$

STATE SPACE MODEL FROM TRANSFER FUNCTION

Apply inverse Laplace transform on both the sides.

$$y(t) = b_n \frac{d^n v(t)}{dt^n} + b_{n-1} \frac{d^{n-1} v(t)}{dt^{n-1}} + \ldots + b_1 \frac{dv(t)}{dt} + b_0 v(t)$$

By substituting the state variables and $y(t) = y$ in the above equation, will get the output equation as,

$$y = b_n \dot{x}_n + b_{n-1} x_n + \ldots + b_1 x_2 + b_0 x_1$$

Substitute, $\dot{x}_n$ value in the above equation.

$$y = b_n(-a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + u) + b_{n-1} x_n + \ldots + b_1 x_2 + b_0 x_1$$

$$y = (b_0 - b_n a_0) x_1 + (b_1 - b_n a_1) x_2 + \ldots + (b_{n-1} - b_n a_{n-1}) x_n + b_n u$$

The state space model is

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

STATE SPACE MODEL FROM TRANSFER FUNCTION

$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = [b_0 - b_n a_0 \quad b_1 - b_n a_1 \quad \dots \quad b_{n-2} - b_n a_{n-2} \quad b_{n-1} - b_n a_{n-1}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

If $b_n = 0$, then,

$$Y = [b_0 \quad b_1 \quad \dots \quad b_{n-2} \quad b_{n-1}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

TRANSFER FUNCTION FROM STATE SPACE MODEL

We know the state space model of a Linear Time-Invariant (LTI) system is -

$$\dot{X}=AX+BU$$

$$Y=CX+DU$$

Apply Laplace Transform on both sides of the state equation.

$$sX(s)=AX(s)+BU(s)$$

$$\Rightarrow(sI-A)X(s)=BU(s)$$

$$\Rightarrow X(s)=(sI-A)-1BU(s)$$

Apply Laplace Transform on both sides of the output equation. $Y(s)=CX(s)+DU(s)$

Substitute, X(s) value in the above equation.

$$\Rightarrow Y(s)=C(sI-A)-1BU(s)+DU(s)$$

$$\Rightarrow Y(s)=[C(sI-A)-1B+D]U(s) \qquad \Rightarrow Y(s)U(s)=C(sI-A)-1B+D$$

TRANSFER FUNCTION FROM STATE SPACE MODEL-EX.

The above equation represents the transfer function of the system.

So, we can calculate the transfer function of the system by using this formula for the system represented in the state space model.

Note –

When D=[0]D=[0], the transfer function will be

$$Y(s)/U(s)=C(sI-A)^{-1}B$$

TRANSFER FUNCTION FROM STATE SPACE MODEL-EX.

Let us calculate the transfer function of the system represented in the state space model as,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad and \quad D = [0]$$

The formula for the transfer function when $D = [0]$ is -

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Substitute, A, B & C matrices in the above equation.

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

TRANSFER FUNCTION FROM STATE SPACE MODEL-EX.

$$\Rightarrow \frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}}{(s+1)s - 1(-1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix}}{s^2 + s + 1} = \frac{1}{s^2 + s + 1}$$

Therefore, the transfer function of the system for the given state space model is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

STATE TRANSITION MATRIX & PROPERTIES

If the system is having initial conditions, then it will produce an output. Since, this output is present even in the absence of input, it is called **zero input response** x ZIR(t)

Mathematically, we can write it as,

$$x_{ZIR}(t) = e^{At} X(0) = L^{-1}\{[sI-A]^{-1} X(0)\}$$

From the above relation, we can write the state transition matrix ϕ(t) as

$$\phi(t) = e^{At} = L^{-1}[sI-A]^{-1}$$

STATE TRANSITION MATRIX & PROPERTIES

So, the zero input response can be obtained by multiplying the state transition matrix ϕ(t)ϕ(t) with the initial conditions matrix.
Following are the properties of the state transition matrix.

- If t=0, then state transition matrix will be equal to an Identity matrix.

$$\phi(0)=I$$

- Inverse of state transition matrix will be same as that of state transition matrix just by replacing 't' by '-t'.

$$\Phi^{-1}(t)=\phi(-t)$$

- If t=t1+t2 , then the corresponding state transition matrix is equal to the multiplication of the two state transition matrices at t=t1 and t=t2.

$$\phi(t1+t2)=\phi(t1)\phi(t2)$$

SUMMARY