# SNS COLLEGE OF TECHNOLOGY

**Coimbatore-45**
**An Autonomous Institution**

Accredited by NBA – AICTE and Accredited by NAAC – UGC with 'A++' Grade
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# DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

## 19ECT212 – CONTROL SYSTEMS

### II YEAR/ IV SEMESTER

### UNIT II – TIME RESPONSE ANALYSIS

## TOPIC 4- IMPULSE AND STEP RESPONSE ANALYSIS OF SECOND ORDER SYSTEMS

# OUTLINE

- REVIEW ABOUT PREVIOUS CLASS
- INTRODUCTION
- SECOND ORDER SYSTEM
- TIME-DOMAIN SPECIFICATION
- TRANSIENT RESPONSE ANALYSIS- RISE TIME,PEAK TIME, PERCENT OVERSHOOT, %OS, SETTING TIME
- ACTIVITY
- UNDERDAMPED-EXAMPLE
- OVERDAMPED RESPONSE
- STEP RESPONSE OF SECOND ORDER SYSTEM
- SUMMARY

# SECOND – ORDER SYSTEM

- ***Second-order systems*** **exhibit a wide range of responses which must be analyzed and described.**
  - **Whereas for a *first-order system*, varying a single parameter changes the speed of response, changes in the parameters of a *second order system* can change the form of the response.**

- ***For example*: a second-order system can display characteristics much like a first-order system or, depending on component values, display damped or *pure oscillations* for its *transient response*.**

# SECOND – ORDER SYSTEM

- A general second-order system is characterized by the following transfer function:

$$G(s) = \frac{b}{s^2 + as + b}$$

- We can re-write the above transfer function in the following form (closed loop transfer function):

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

# SECOND – ORDER SYSTEM

$\omega_n$ $(\omega_n = \sqrt{b})$ - referred to as *the un-damped natural frequency* of the second order system, which is the frequency of oscillation of the system without damping.

$\zeta$ $(\zeta = \frac{a}{2\sqrt{b}})$ - referred to as *the damping ratio* of the second order system, which is a measure of the degree of resistance to change in the system output.

Poles;

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

Poles are complex if $\zeta < 1$!

# SECOND – ORDER SYSTEM

- According the value of ζ, a second-order system can be set into one of the four categories:

1. *Overdamped* - when the system has two real distinct poles ($\zeta > 1$).
2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)
3. *Undamped* - when the system has two imaginary poles ($\zeta = 0$).
4. *Critically damped* - when the system has two real but equal poles ($\zeta = 1$).

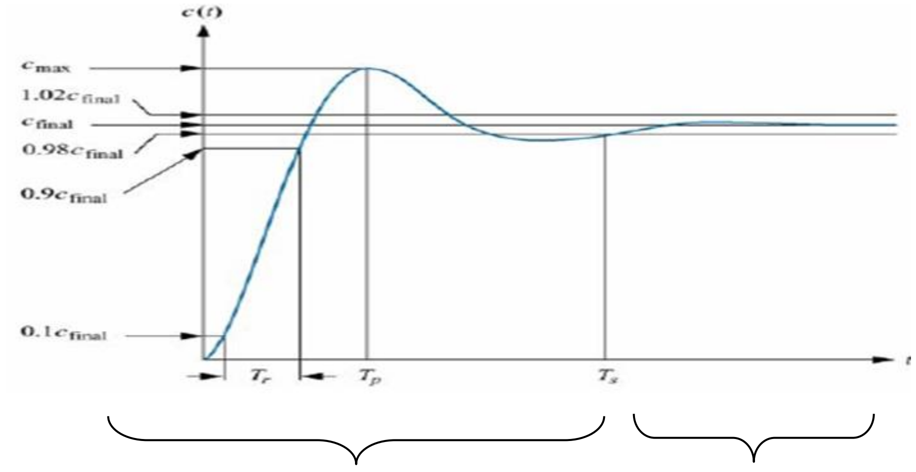
# TIME-DOMAIN SPECIFICATION

Given that the closed loop TF

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^{\;2}}{s^2 + 2\varsigma\omega_n s + \omega_n^{\;2}}$$

The system (2nd order system) is parameterized by $\varsigma$ and $\omega_n$

For $0 < \varsigma < 1$ and $\omega_n > 0$, we like to investigate its response due to a unit step input




Two types of responses that are of interest:
(A) Transient response
(B) Steady state response

# (A) For transient response 4 specifications:

(a) $T_r$ – rise time = $\dfrac{\pi - \theta}{\omega_n \sqrt{1-\varsigma^2}}$

(b) $T_p$ – peak time = $\dfrac{\pi}{\omega_n \sqrt{1-\varsigma^2}}$

(c) %MP – percentage maximum overshoot = $e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}} x100\%$

(d) $T_s$ – settling time (2% error) = $\dfrac{4}{\varsigma\omega_n}$

# (B) Steady State Response

(a) Steady State error-NEXT TOPIC…

# Question : How are the performance related to ς and $\omega_n$ ?

- Given a step input, i.e., *R*(s) =1/*s,* then the system output (or step response) is;

$$C(s) = R(s)G(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- Taking inverse Laplace transform, we have the step response;

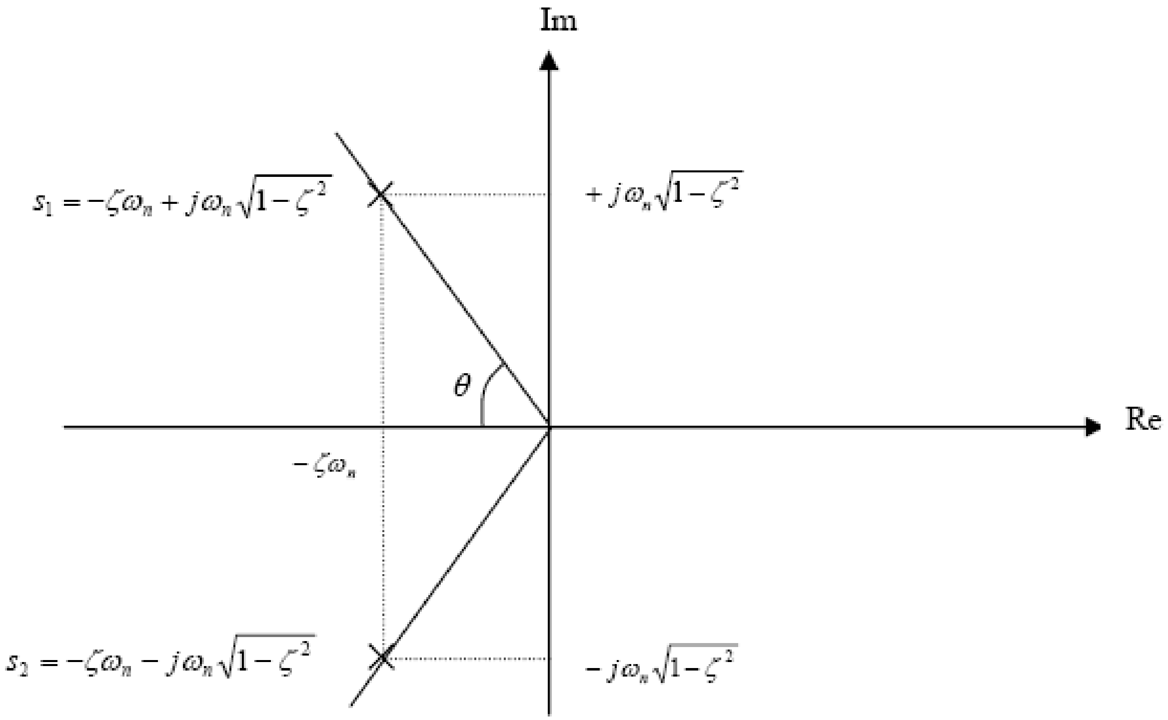
$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \theta\right)$$

Where;

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \text{or} \quad \theta = \cos^{-1}(\xi)$$

# SECOND – ORDER SYSTEM

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$




$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\varsigma^2}}{\varsigma}\right)$$

Mapping the poles into s-plane

Lets re-write the equation for c(t):

Let: $$\beta = \sqrt{1-\xi^2}$$

and

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

} Damped natural frequency $\omega_n > \omega_d$

Thus:

$$c(t) = 1 - \frac{1}{\beta} e^{-\xi\omega_n t} \sin(\omega_d t + \theta)$$

where $\theta = \cos^{-1}(\xi)$

# TRANSIENT RESPONSE ANALYSIS

*1) Rise time, $T_r$*. Time the response takes to rise from 0 to 100%

$$c(t)\big|_{t=T_r} = 1 - \underbrace{\frac{1}{\beta}e^{-\xi\omega_n t}}_{\neq 0}\underbrace{\sin(\omega_d t + \theta)}_{=0} = 1$$

$$\sin(\omega_d T_r + \theta) = 0$$

$$\omega_d T_r + \theta = \sin^{-1}(0) = \pi$$

$$T_r = \frac{\pi - \theta}{\omega_n\sqrt{1-\xi^2}}$$

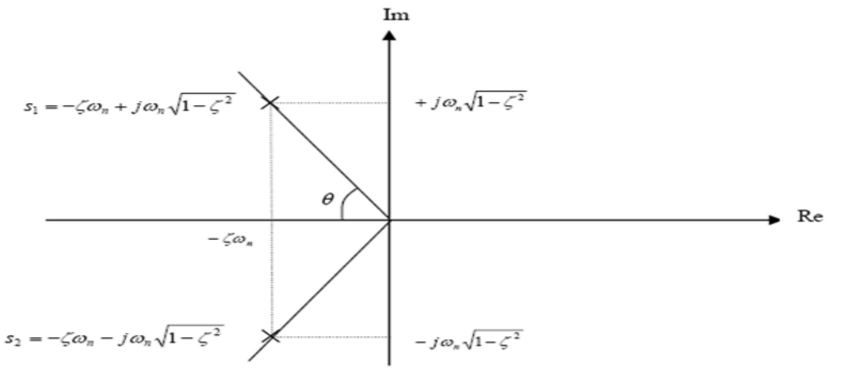
*2) Peak time, $T_p$* - The peak time is the time required for the response to reach the first peak, which is given by;

$$\dot{c}(t)\Big|_{t=T_p} = 0$$

$$\dot{c}(t)\Big|_{t=T_p} = -\frac{1}{\beta}(-\varsigma\omega_n)e^{-\varsigma\omega_n t}\sin(\omega_d t+\theta) - \frac{1}{\beta}e^{-\varsigma\omega_n t}\cos(\omega_d t+\theta)\left[\omega_n\sqrt{1-\varsigma^2}\right] = 0$$

$$\frac{\varsigma\omega_n}{\beta}e^{-\varsigma\omega_n T_p}\sin(\omega_d T_p+\theta) = \frac{\left[\omega_n\sqrt{1-\varsigma^2}\right]}{\beta}e^{-\varsigma\omega_n T_p}\cos(\omega_d T_p+\theta)$$

$$\tan(\omega_d T_p+\theta) = \frac{\sqrt{1-\varsigma^2}}{\varsigma}$$




$$\tan\theta = \frac{\sqrt{1-\varsigma}}{\varsigma}$$

# TRANSIENT RESPONSE ANALYSIS

*3) Percent overshoot, %OS* - The percent overshoot is defined as the amount that the waveform at the peak time overshoots the steady-state value, which is expressed as a percentage of the steady-state value.

$$\%MP \equiv \frac{C(T_p) - C(\infty)}{C(\infty)} x100\%$$

OR

$$\%OS = \frac{C\max - Cfinal}{Cfinal} \; x\; 100$$

We know that $\tan(\theta) = \tan(\pi + \theta)$

So, $\tan(\omega_d T_p + \theta) = \tan(\pi + \theta)$

From this expression:

$$\omega_d T_p + \theta = \pi + \theta$$

$$\omega_d T_p = \pi$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\varsigma^2}}$$

$$\frac{C(T_p)-1}{1}x100\% = -\frac{1}{\beta}e^{-\xi\omega_n t}\sin(\omega_d t+\theta)x100\%$$

$$= -\frac{1}{\beta}e^{-\xi\omega_n\left[\frac{\pi}{\omega_n\sqrt{1-\varsigma^2}}\right]}\sin\left(\omega_d\left(\frac{\pi}{\omega_d}\right)+\theta\right)x100\%$$

$$= -\frac{1}{\beta}e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}\sin(\pi+\theta)x100\%$$

$$= \frac{\sin(\theta)}{\beta}e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}x100\% = e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}x100\%$$

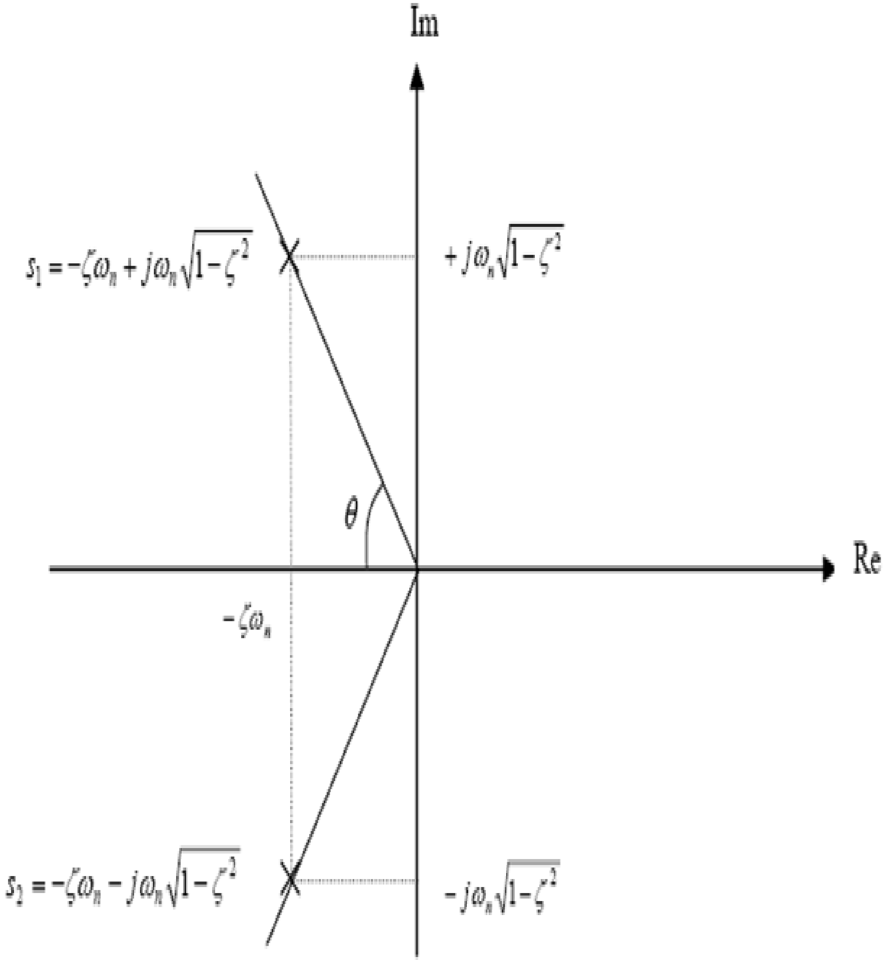
$$\beta = \sqrt{1-\xi^2}$$

$$\sin\theta = \sqrt{1-\varsigma^2}$$

# ACTIVITY-GD

Therefore,

$$\%MP = e^{-\frac{\pi \varsigma}{\sqrt{1-\varsigma^2}}} x100\%$$




- For given $\%OS$, the damping ratio can be solved from the above equation;

$$\varsigma = \frac{-\ln(\%MP/100)}{\sqrt{\pi^2 + \ln^2(\%MP/100)}}$$

# TRANSIENT RESPONSE ANALYSIS

*4) Setting time, $T_s$* - The settling time is the time required for the amplitude of the sinusoid to decay to 2% of the steady-state value.

To find $T_s$, we must find the time for which c(t) reaches & stays within ±2% of the steady state value, $c_{final.}$ The settling time is the time it takes for the amplitude of the decaying sinusoid in c(t) to reach 0.02, or

$$e^{-\varsigma\omega_n T_s}\frac{1}{\sqrt{1-\varsigma^2}} = 0.02$$

Thus,

$$T_s = \frac{4}{\varsigma\omega_n}$$

# UNDERDAMPED

Example 2: *Find the natural frequency and damping ratio for the system with transfer function*

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Solution:

**Compare with general TF**

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- $\omega_n = 6$
- $\xi = 0.35$

# UNDERDAMPED

Example 3: Given the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

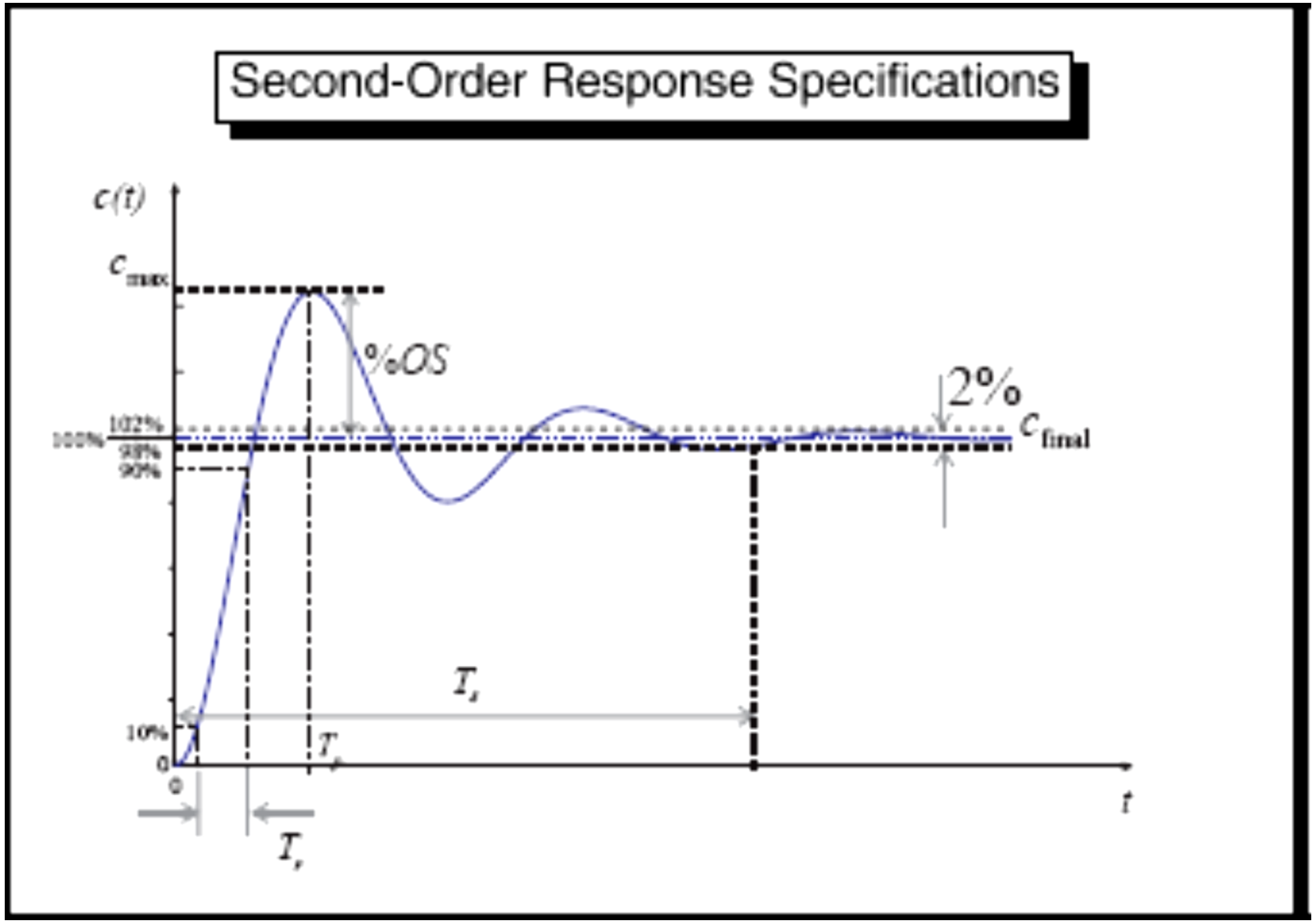
$$find \quad T_s, \%OS, T_p$$

Solution:

$$\omega_n = 10 \qquad \xi = 0.75$$

$$T_s = 0.533s, \ \%OS = 2.838\%, \ T_p = 0.475s$$

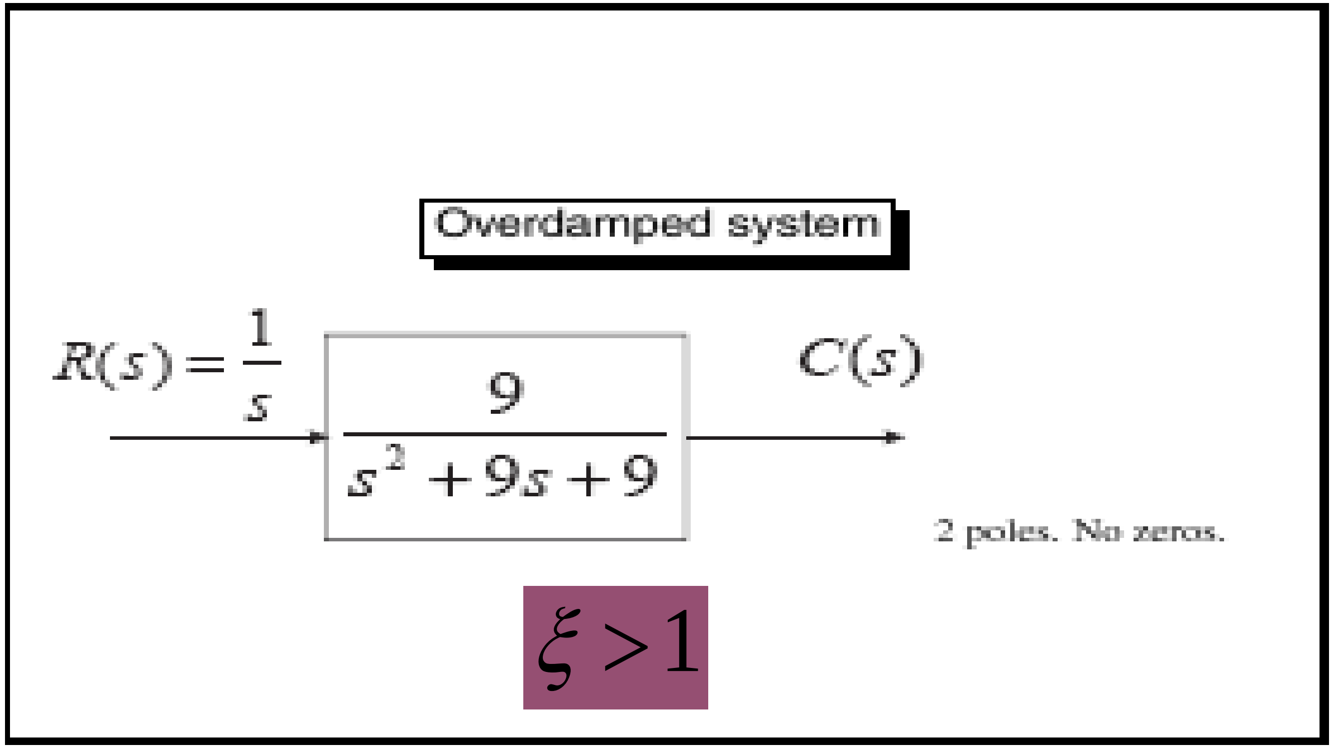
# UNDERDAMPED



# OVERDAMPED RESPONSE

$$G(s) = \frac{b}{s^2 + as + b}$$

a = 9

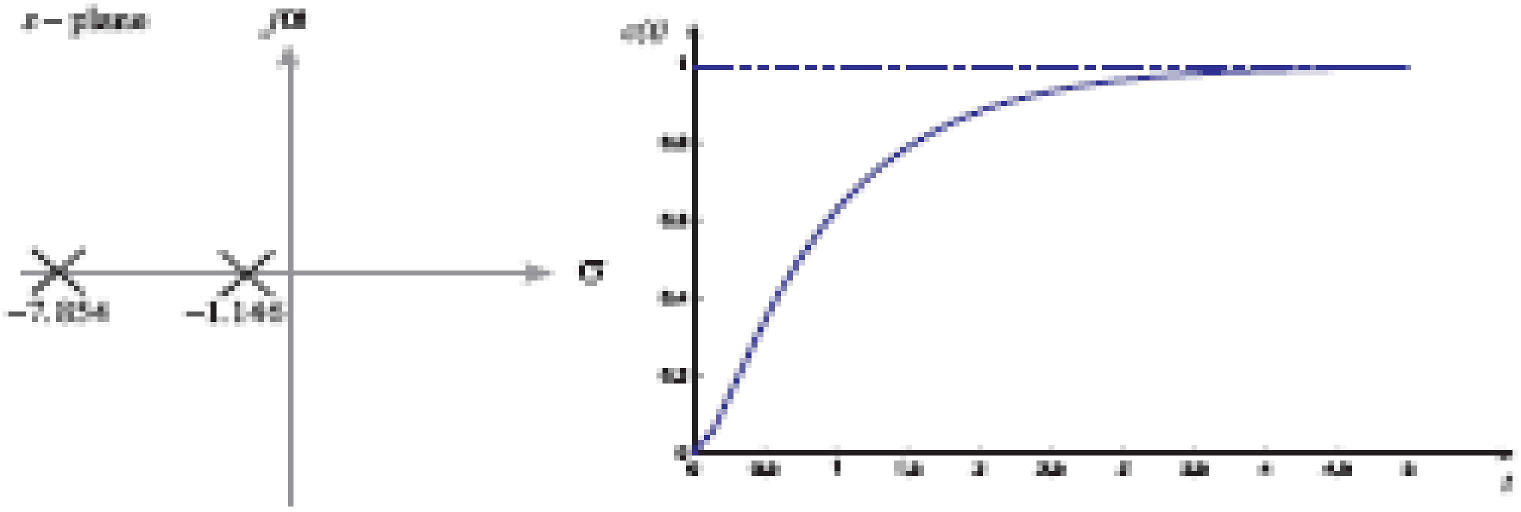



$$C(s) = \frac{9}{s(s^2 + 9s + 9)} = \frac{9}{s(s + 7.854)(s + 1.146)}$$

s= 0; s = -7.854; s = -1.146 *( two real poles)*

$$c(t) = K_1 + K_2 e^{-7.854t} + K_3 e^{-1.146t}$$

Overdamped response

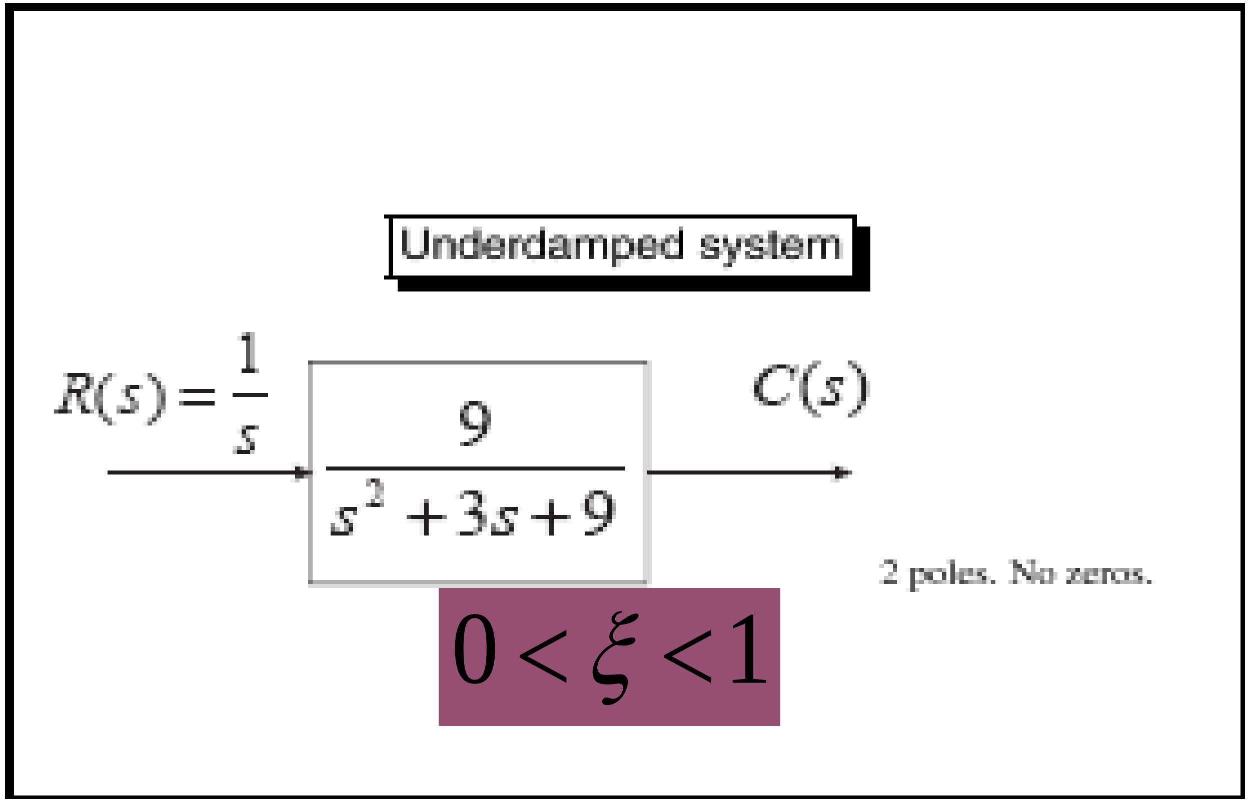



OVERDAMPED RESPONSE !!!

# Underdamped Response

$$G(s) = \frac{b}{s^2 + as + b}$$

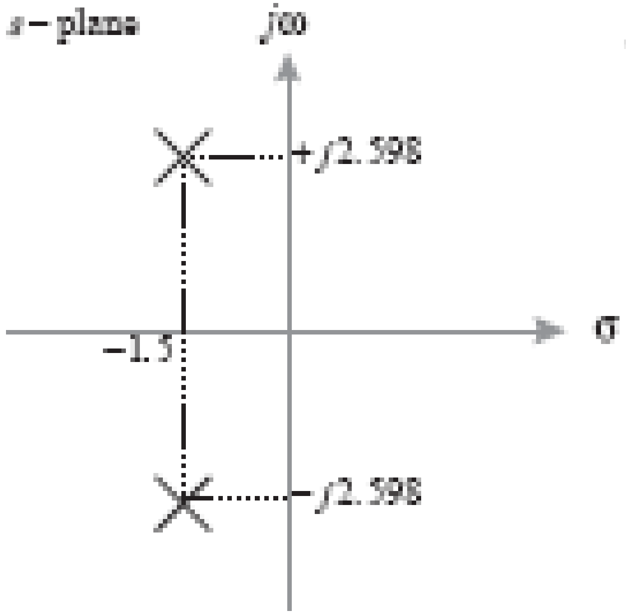
a = 3

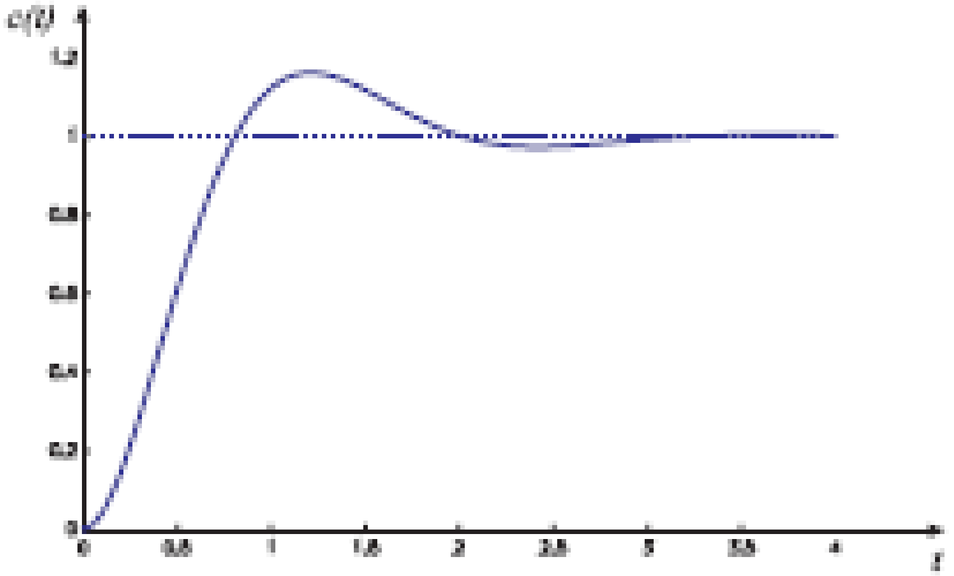



$$c(t) = K_1 + e^{-1.5t}(K_2 \cos 2.598t + K_3 \sin 2.598t)$$

s = 0; s = -1.5 ± j2.598 *( two complex poles)*

# Underdamped response



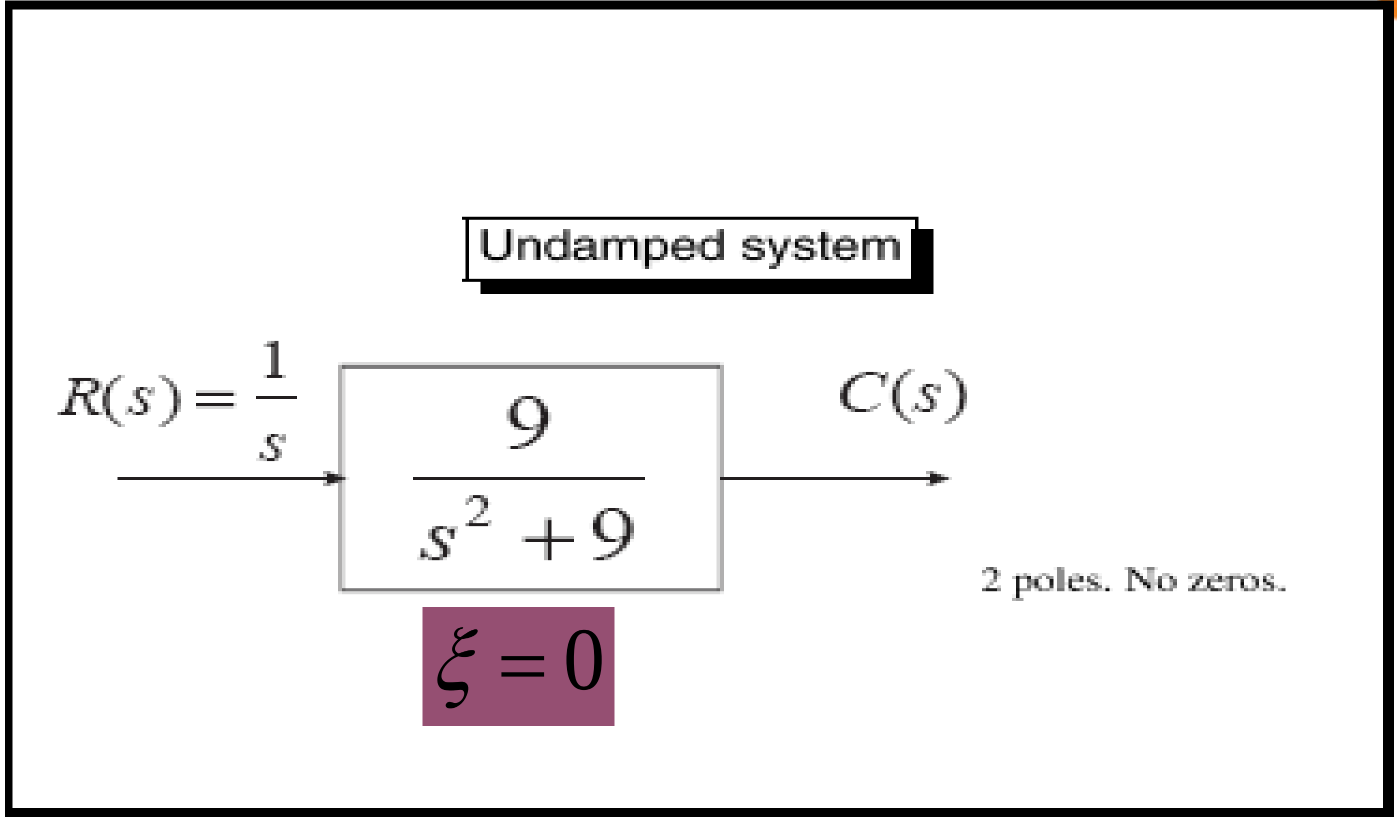




## UNDERDAMPED RESPONSE !!!

# Undamped Response

$$G(s) = \frac{b}{s^2 + as + b}$$

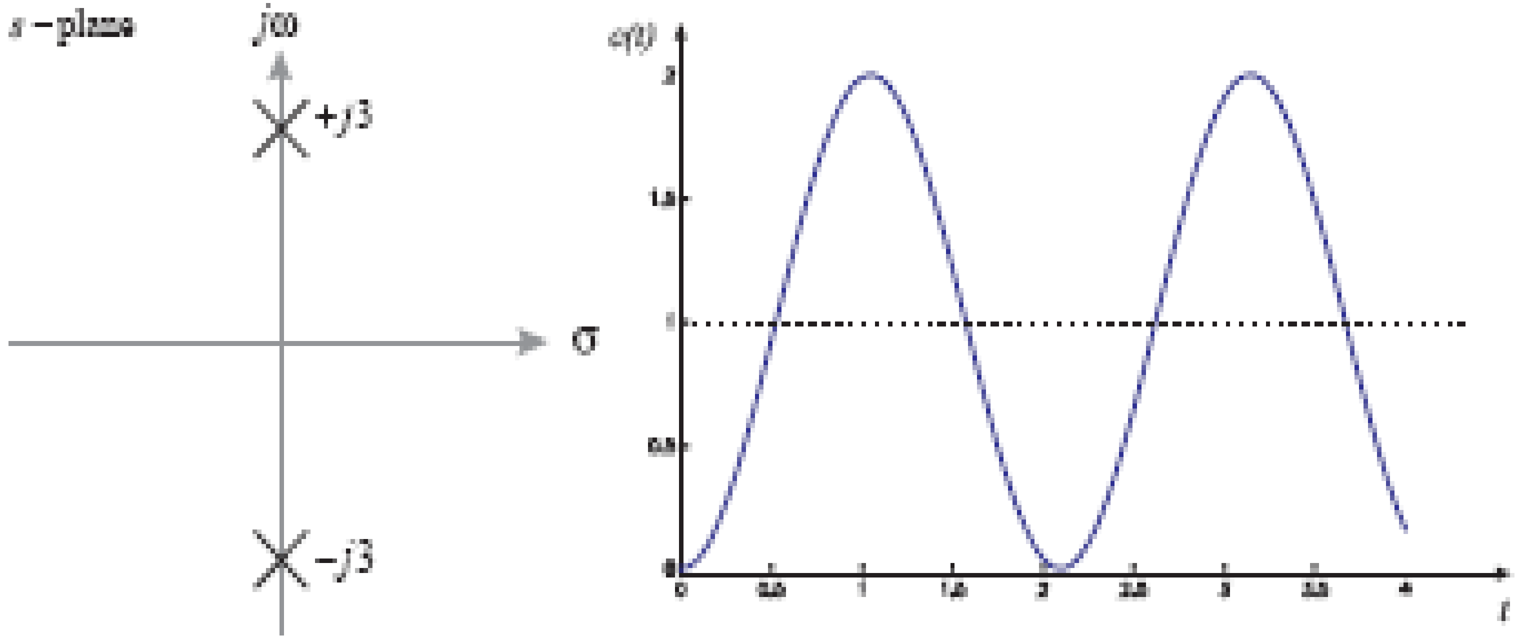
a = 0




$$c(t) = K_1 + K_2 \cos 3t$$

s = 0; s = ± j3 *( two imaginary poles)*

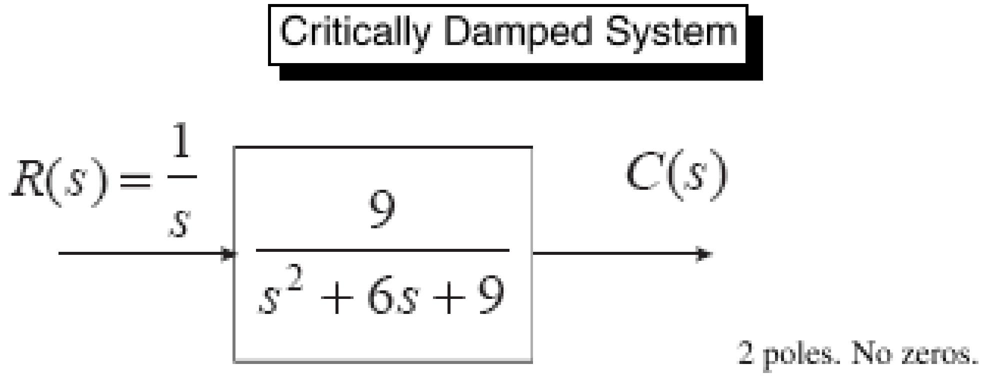
## Undamped response



# Critically Damped System

$$G(s) = \frac{b}{s^2 + as + b}$$

a = 6

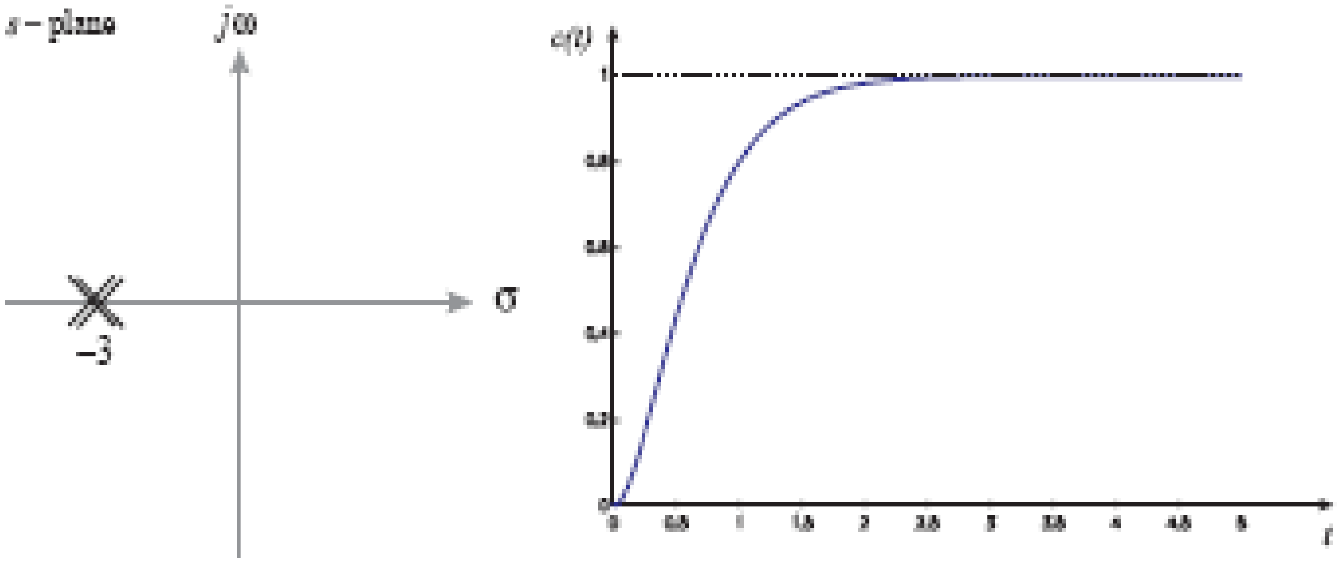



$$\xi = 1$$

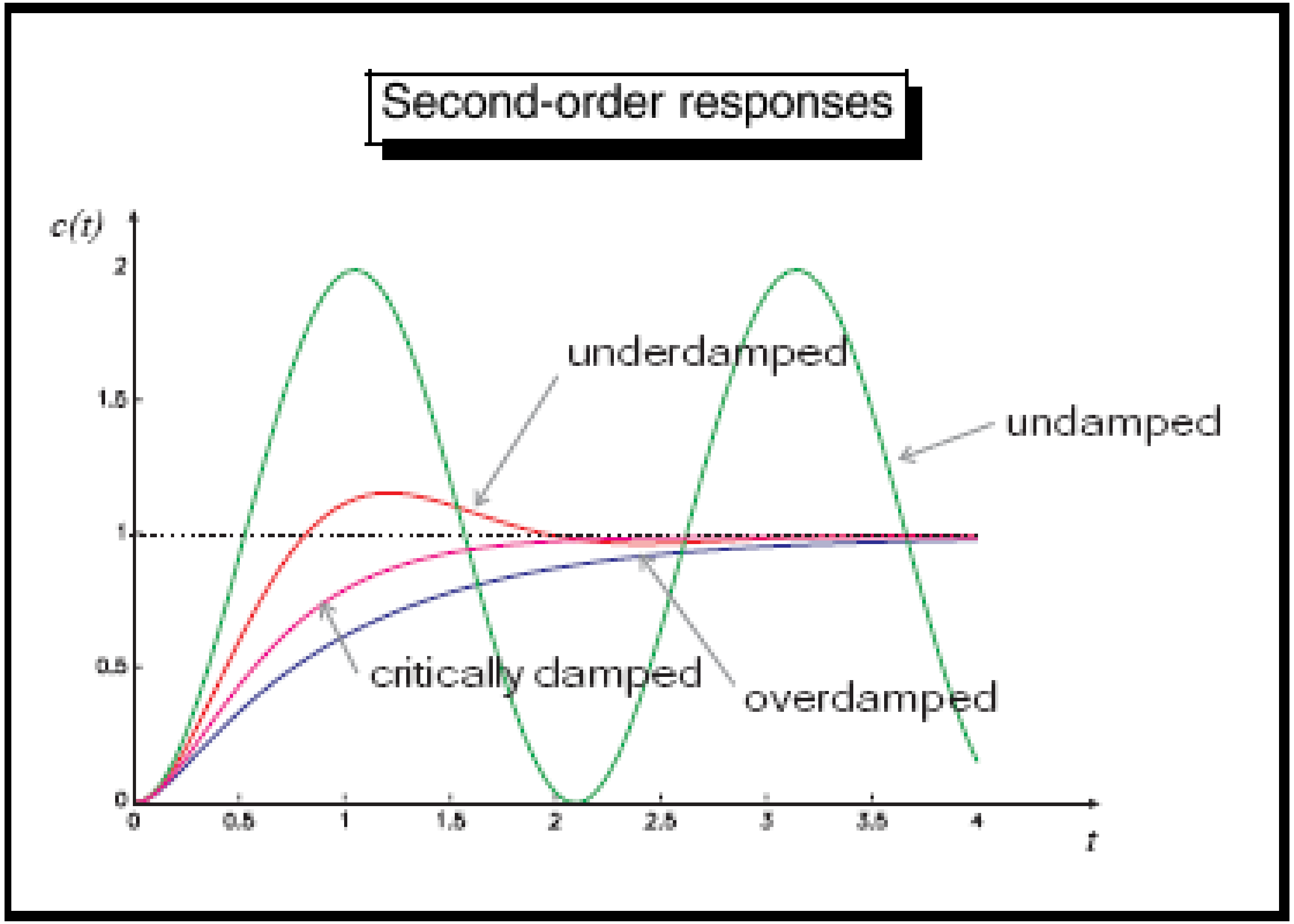
$$c(t) = K_1 + K_2 e^{-3t} + K_3 t e^{-3t}$$

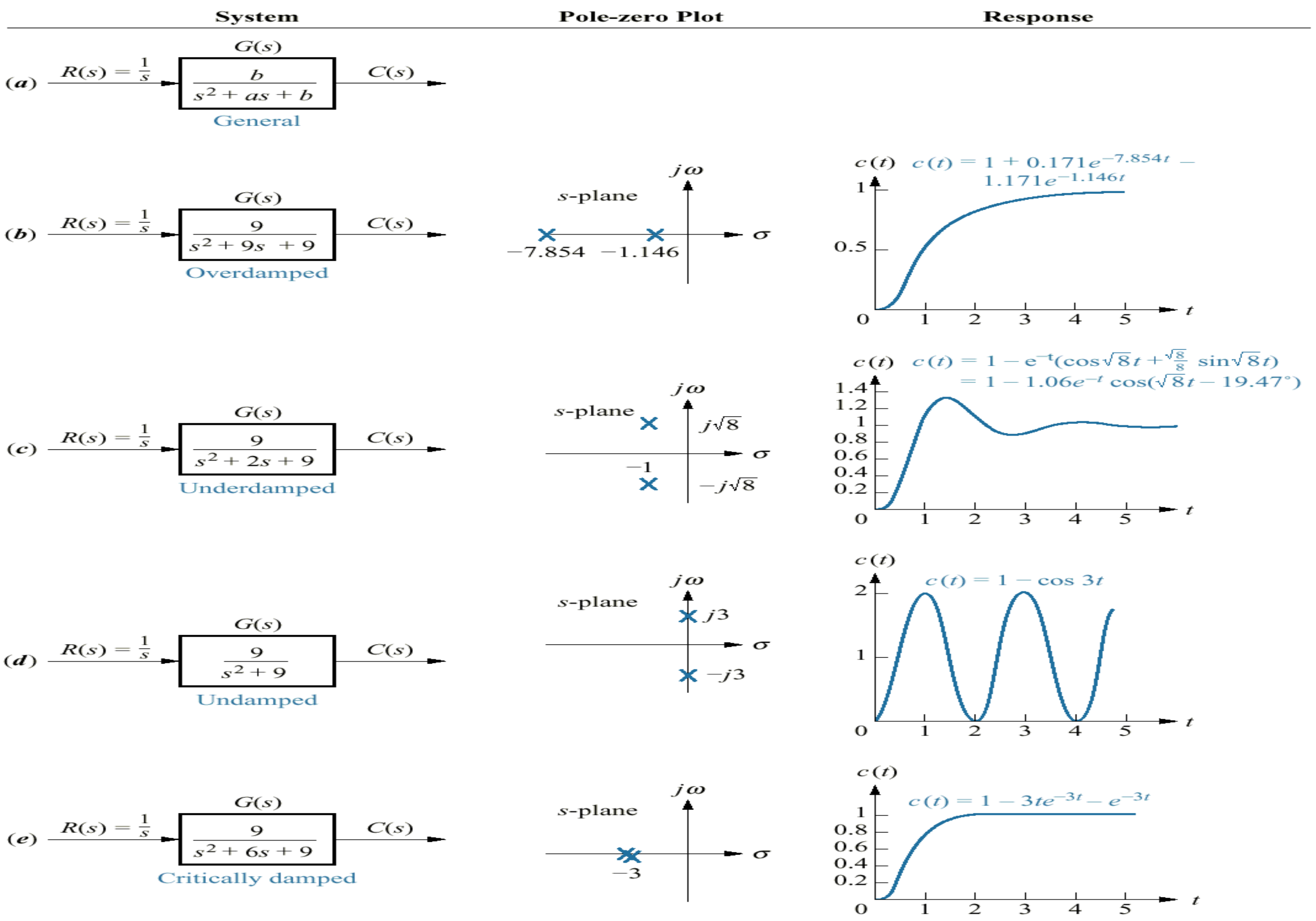
S = 0; s = -3,-3 *( two real and equal poles)*

# Critically Damped Response

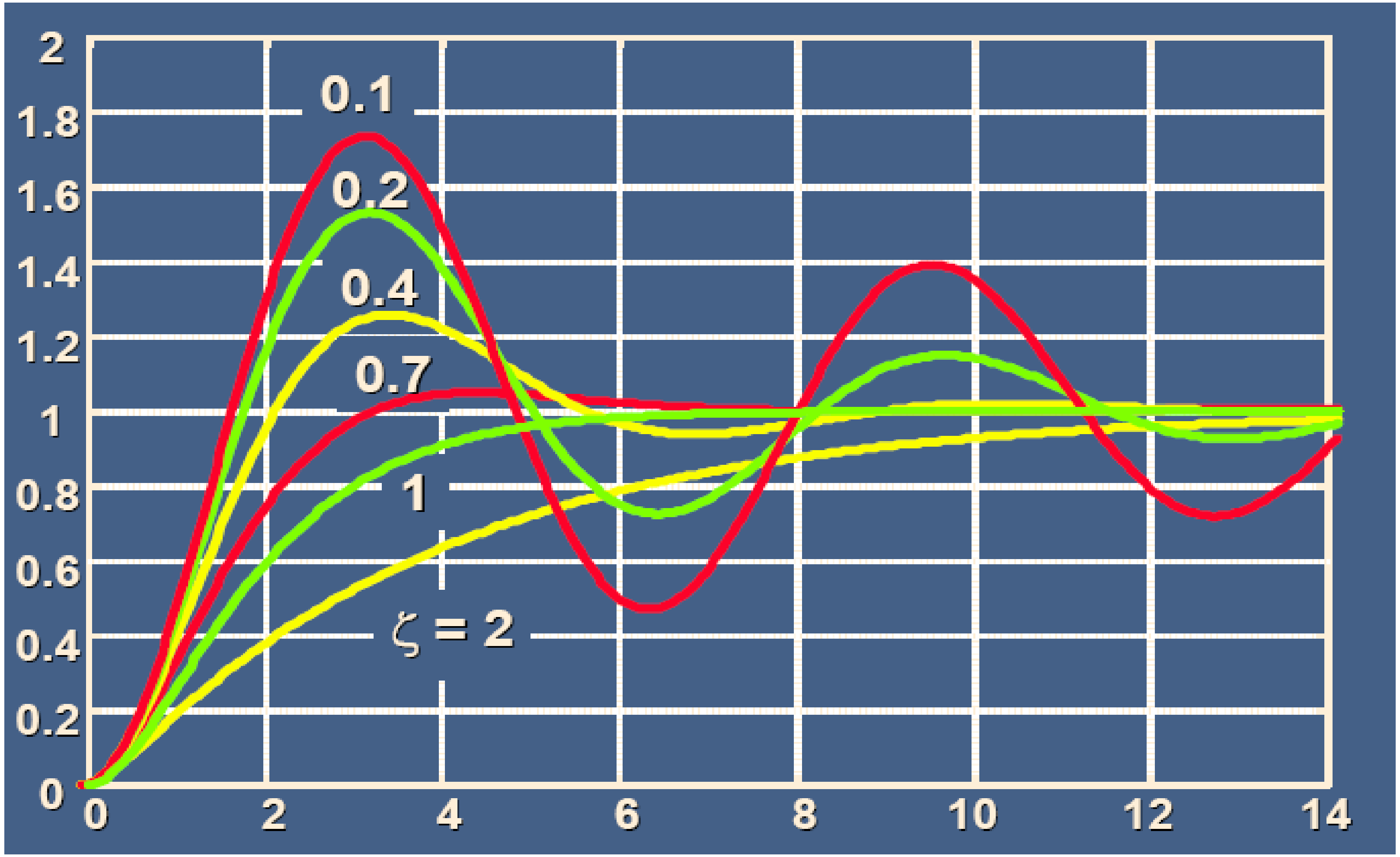


# SECOND – ORDER SYSTEM



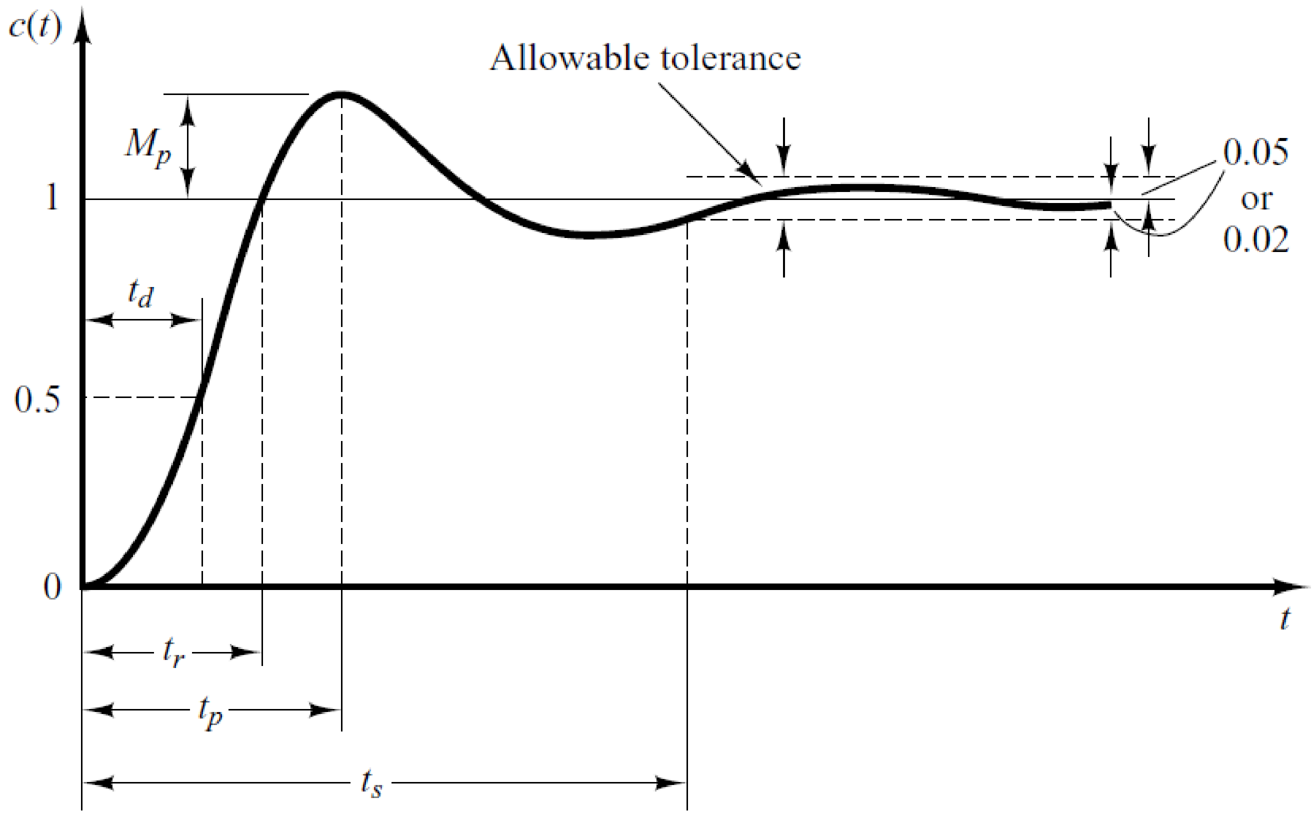

System
Pole-zero Plot
Response
(a)
R(s) = 1/s
G(s)
b / (s² + as + b)
C(s)
General
(b)
R(s) = 1/s
G(s)
9 / (s² + 9s + 9)
C(s)
Overdamped
s-plane
jω
σ
−7.854
−1.146
c(t) = 1 + 0.171e^(−7.854t) − 1.171e^(−1.146t)
(c)
R(s) = 1/s
G(s)
9 / (s² + 2s + 9)
C(s)
Underdamped
s-plane
jω
σ
j√8
−1
−j√8
c(t) = 1 − e^(−t)(cos√8t + (√8/8) sin√8t)
= 1 − 1.06e^(−t) cos(√8t − 19.47°)
(d)
R(s) = 1/s
G(s)
9 / (s² + 9)
C(s)
Undamped
s-plane
jω
σ
j3
−j3
c(t) = 1 − cos 3t
(e)
R(s) = 1/s
G(s)
9 / (s² + 6s + 9)
C(s)
Critically damped
s-plane
jω
σ
−3
c(t) = 1 − 3te^(−3t) − e^(−3t)
c(t)
t

# EFFECT OF DIFFERENT DAMPING RATIO, ξ



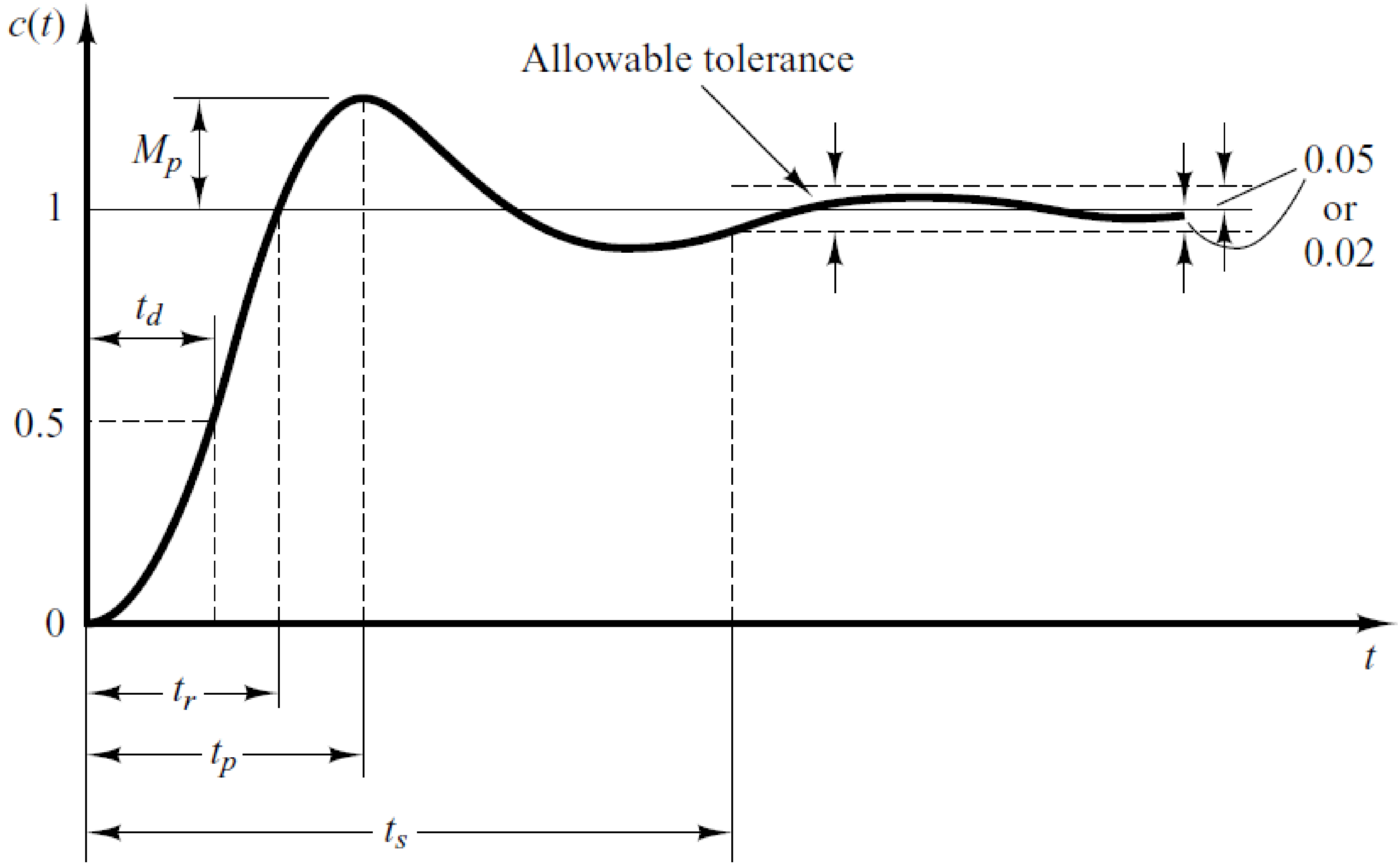
# UNDERDAMPED SYSTEM

For 0<  <1 and $\omega_n > 0$, the 2nd order system's response due to a unit step input:
**Important timing characteristics:** delay time, rise time, peak time, maximum overshoot, and settling time.



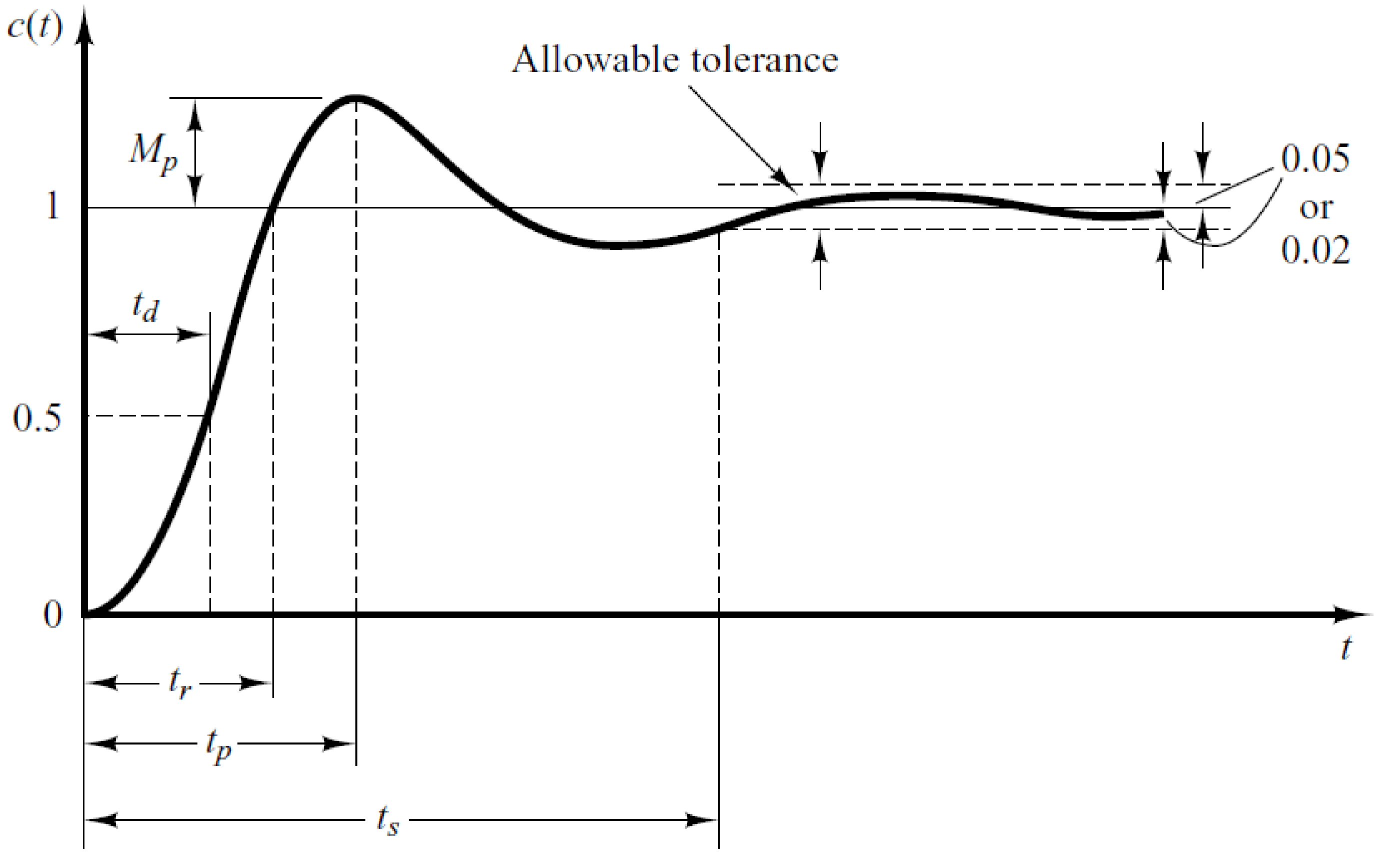
# DELAY TIME

- The delay ($t_d$) time is the time required for the response to reach half the final value the very first time.



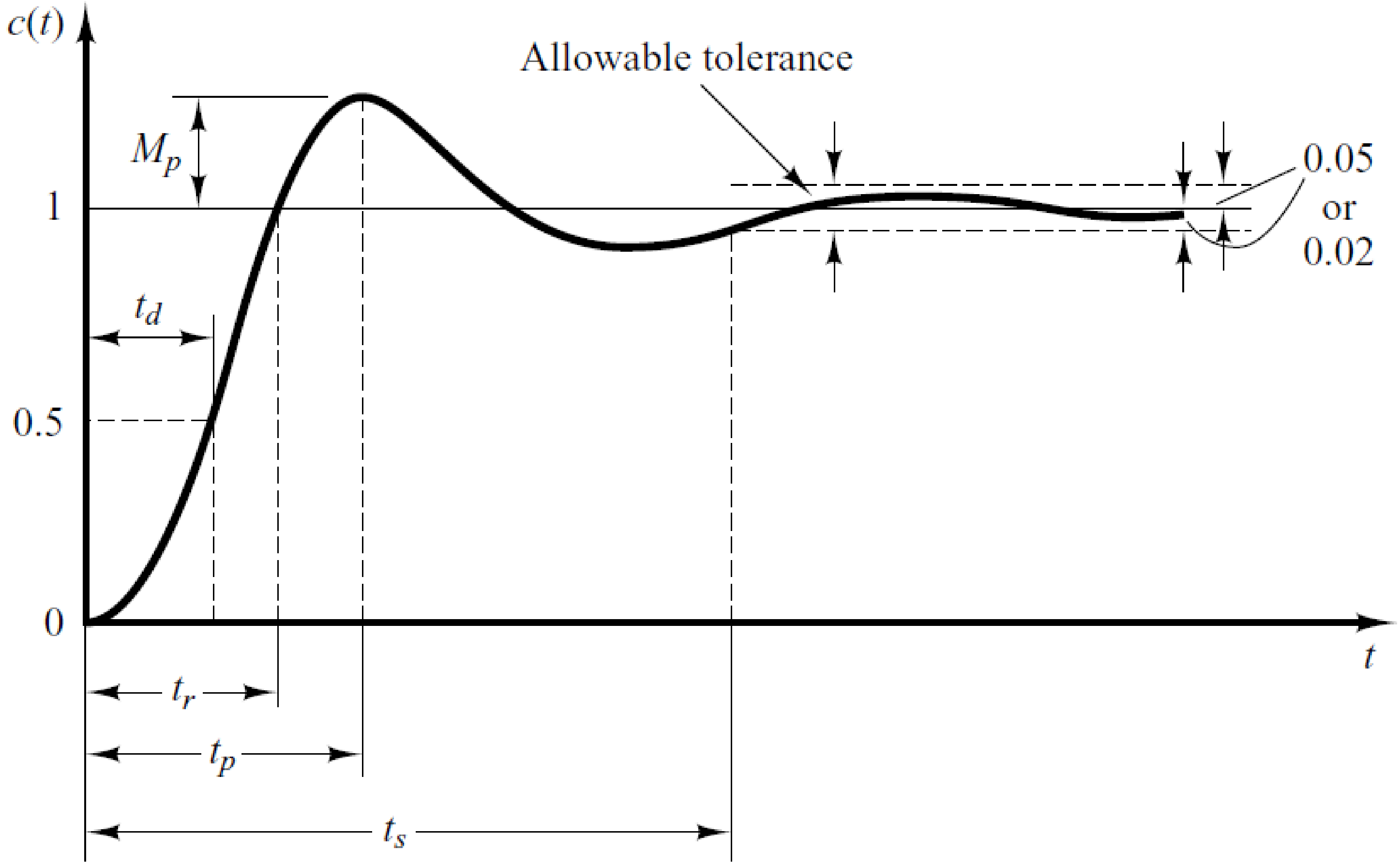
# Rise Time

- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.



# PEAK TIME

- The peak time is the time required for the response to reach the first peak of the overshoot.



# Maximum Overshoot

The maximum overshoot is the maximum peak value of the response curve measured from unity.

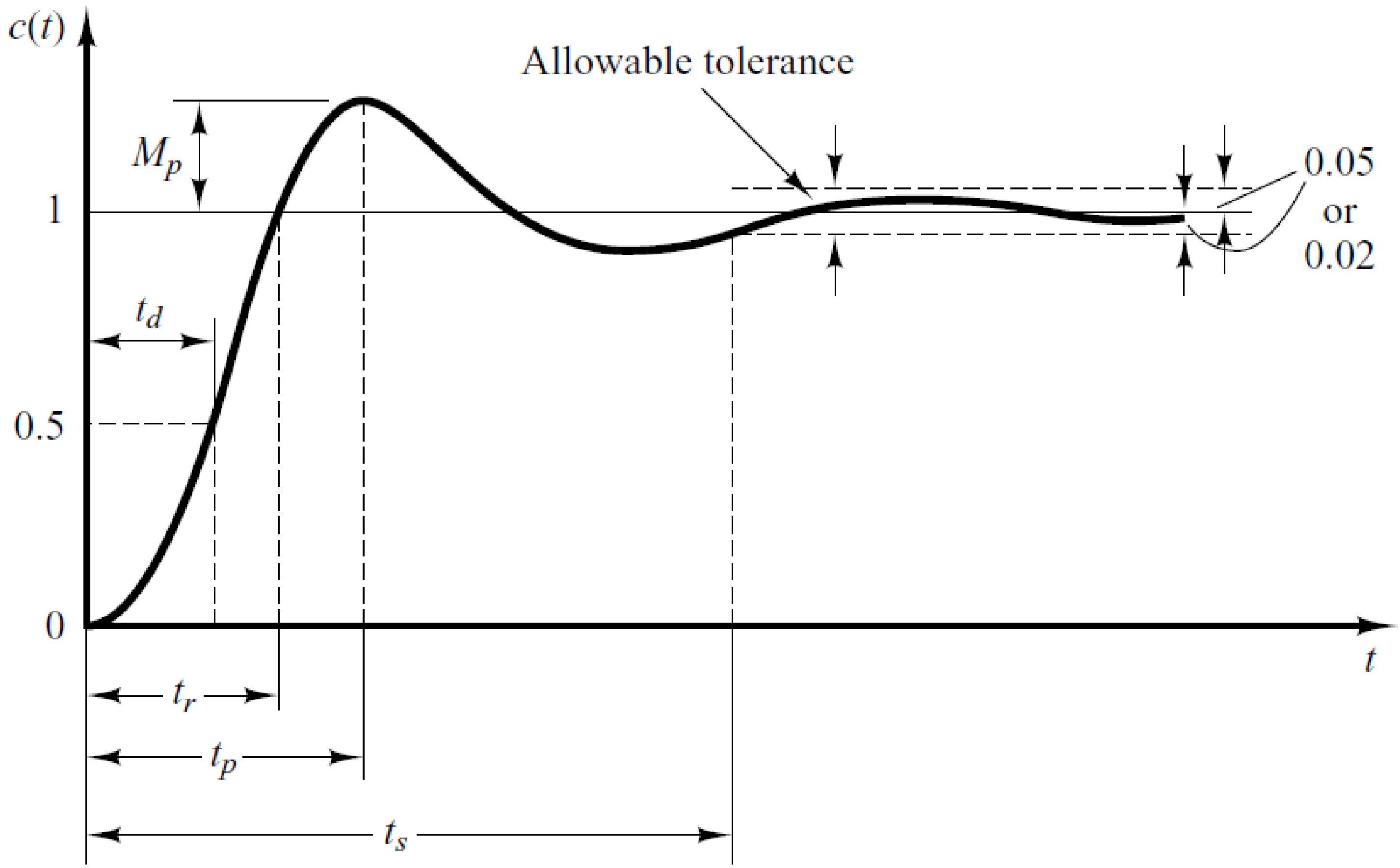
If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

# Settling Time

- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



# STEP RESPONSE OF UNDERDAMPED SYSTEM

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- The partial fraction expansion of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$(s + 2\zeta\omega_n)^2$ $\qquad$ $\omega_n^2\left(1 - \zeta^2\right)$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1 - \zeta^2\right)}$$

# STEP RESPONSE OF UNDERDAMPED SYSTEM

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

- Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- Where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ , is the frequency of transient oscillations and is called damped natural frequency.

- The inverse Laplace transform of above equation can be obtained easily if C(s) is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

# STEP RESPONSE OF UNDERDAMPED SYSTEM

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

# STEP RESPONSE OF UNDERDAMPED SYSTEM

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- When $\zeta = 0$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

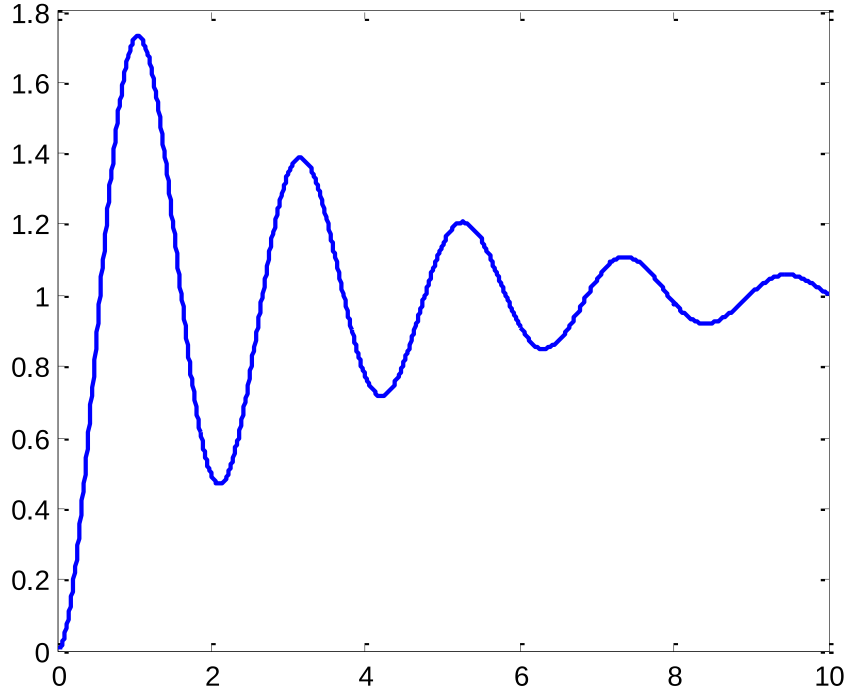
$$= \omega_n$$

$$c(t) = 1 - \cos\omega_n t$$

# STEP RESPONSE OF UNDERDAMPED SYSTEM

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

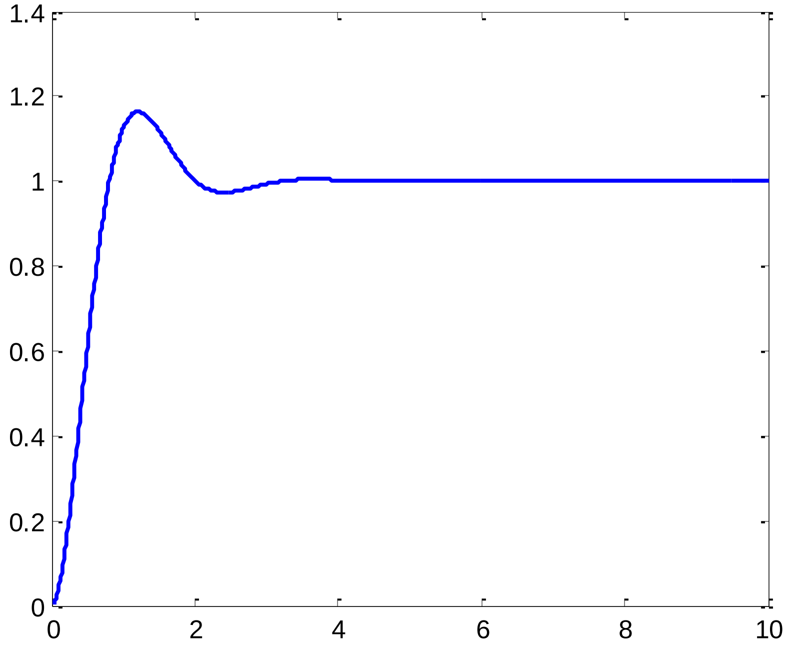
if $\zeta = 0.1$ and $\omega_n = 3$



# STEP RESPONSE OF UNDERDAMPED SYSTEM

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

if $\zeta = 0.5$ and $\omega_n = 3$



# SUMMARY