Type : 2 RHS = $\cos(ax+by)$ or $\sin(ax+by)$

Replace $D^2 \to -a^2$

$DD' \to -ab$

$D'^2 \to -b^2$

1. Solve: $(D^2 - 2DD' + D'^2)z = \cos(x-3y)$

Auxillary Equation is $m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$ (equal)

$\therefore$ CF is $f_1(y+x) + xf_2(y+x)$

$PI = \frac{1}{D^2 - 2DD' + D'^2}\cos(x-3y)$

$a = 1$, $b = -3$

$D^2 \to -a^2 = -1$

$DD' = -ab = -(+1)(-3) = 3$

$D'^2 = -b^2 = -(-3)^2 = -9$

$= \frac{1}{-1-2(3)-9}\cos(x-3y)$

$= \frac{-1}{16}\cos(x-3y)$

$z = CF + PI$

$= f_1(y+x) + xf_2(y+x) - \frac{1}{16}\cos(x-3y)$

2. Solve: $(D^2 - 4D'^2)z = \sin(2x+y)$

AE is $m^2 - 4 = 0 \Rightarrow m^2 = 2^2$ $m = \pm 2$

$m = 2, -2$ (Different)

$a = 2$

$b = 1$

$D^2 = -a^2 = -2^2 = -4$

$CF = f_1(y-2x) + f_2(y+2x)$

$PI = \frac{1}{D^2 - 4D'^2}\sin(2x+y) = \frac{1}{-4-4(-1)}\sin(2x+y)$

$DD' \to -ab = -2$

$D'^2 = -b^2 = -1^2$

$= \frac{x\sin(2x+y)}{2D} = \frac{x\,2D(\sin(2x+y))}{(4D^2)} = \frac{4x}{4(-4)}(-\cos(2x+y))$

$$PI = \frac{-x}{4}\cos(2x+y)$$

The solution is $z = CF + PI$

$$= f_1(y-2x) + f_2(y+2x) - \frac{x}{4}\cos(2x+y)$$

3. Find the PI of $(D^2 - 3DD' + D'^2)z = \sin x \cos y$.

$$PI = \frac{1}{D^2 - 3DD' + D'^2}\sin x \cos y$$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

Given: $(D^2 - 3DD' + D'^2)z = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 3DD' + D'^2}\sin(x+y) + \frac{1}{D^2 - 3DD' + D'^2}\sin(x-y)\right]$$

$$= \frac{1}{2}[PI_1 + PI_2] \rightarrow (1)$$

$$PI_1 = \frac{1}{D^2 - 3DD' + D'^2}\sin(x+y)$$

$a = 1,\ b = 1$
$D^2 \Rightarrow -a^2 = -1$
$DD' \Rightarrow -ab = -1$
$D'^2 \Rightarrow -b^2 = -1$

$$= \frac{1}{-1 - 3(-1) + (-1)}\sin(x+y)$$

$$= \frac{1}{-1+3-1}\sin(x+y) = \sin(x+y)$$

$$PI_2 = \frac{1}{D^2 - 3DD' + D'^2}\sin(x-y) = \frac{1}{-1 - 3(1) - 1}\sin(x-y)$$

$$= \frac{-1}{5}\sin(x-y)$$

$$PI = \frac{1}{2}\left[\sin(x+y) - \frac{1}{5}\sin(x-y)\right]$$

H.W: 1. Find the PI of $(D^2+3DD'-4D'^2)z = \sin y$

2. Find the PI of $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = \sin(3x+2y)$

Type 3: $x^m y^n$

1. Solve $(D^2-4DD'+4D'^2)z = xy$

AE is $m^2-4m+4=0 \Rightarrow (m-2)(m-2)=0$

$m = 2, 2.$

$CF = f_1(y+2x) + x f_2(y+2x)$

$PI = \frac{1}{D^2-4DD'+4D'^2} xy = \frac{1}{D^2\left[1-\frac{4DD'}{D^2}+\frac{4D'^2}{D^2}\right]} xy$

$= \frac{1}{D^2}\left[1-\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)\right]^{-1} xy$

$= \frac{1}{D^2}\left[1+\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)+\dots\right] xy \qquad \left[\because (1-x)^{-1} = 1+x+x^2+\dots\right.$

$= \frac{1}{D^2}\left[xy + \frac{4D'}{D}(xy) - 0\right]$

$= \frac{1}{D^2}\left[xy + \frac{4}{D}x\right] = \frac{1}{D^2}xy + \frac{4}{D^3}x$

$\left[\begin{array}{l} \frac{1}{D^2}xy \xrightarrow{1st} \frac{1}{D}\frac{x^2}{2}y \xrightarrow{2nd} \frac{x^3}{6}y \\ \frac{1}{D^3}x \rightarrow \frac{1}{D^2}\frac{x^2}{2} \rightarrow \frac{1}{D}\frac{x^3}{6} \rightarrow \frac{x^4}{24} \end{array}\right.$

$$= \frac{x^3}{6}y + \frac{4x^4}{24}$$

$$= \frac{x^3}{6}y + \frac{x^4}{6}$$

$\therefore$ The solution is $z = CF + PI$

$$z = f_1(y+2x) + x f_2(y+2x) + \frac{1}{6}\left[x^3y + x^4\right]$$

2) Find the PI of $(D^2 - DD' - 2D'^2)z = 2x+3y$

$$PI = \frac{1}{D^2 - DD' - 2D'^2}(2x+3y)$$

$$= \frac{1}{D^2\left[1 - \frac{DD'}{D^2} - \frac{2D'^2}{D^2}\right]}(2x+3y) = \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} - \frac{2D'^2}{D^2}\right)\right]^{-1}(2x+3y)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - \frac{2D'^2}{D^2}\right) + \dots\right](2x+3y)$$

$$= \frac{1}{D^2}\left[2x+3y + \frac{D'}{D}(2x+3y)\right]$$

$$= \frac{1}{D^2}\left[2x+3y + \frac{1}{D}(3)\right] = \frac{1}{D^2}(2x+3y) + \frac{1}{D^3}(3)$$

$$\frac{1}{D^2}(2x+3y) \longrightarrow \frac{1}{D}\left(\frac{2x^2}{2} + 3xy\right) \longrightarrow \frac{x^3}{3} + \frac{3x^2y}{2}$$

$$\frac{1}{D^3}(3) \longrightarrow \frac{1}{D^2}3x \longrightarrow \frac{1}{D}\frac{3x^2}{2} \longrightarrow \frac{x^3}{2}$$

$$PI = \frac{x^3}{3} + \frac{3x^2y}{2} + \frac{x^2}{2}$$

RHS $\Rightarrow e^{ax+by} + \sin(ax+by)$ (or) $e^{ax+by} + \cos(ax+by)$

1) Solve: $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x-y)$

AE is $m^2 - m - 20 = 0$

$(m-5)(m+4) = 0 \Rightarrow m = 5, -4$

$CF = f_1(y-4x) + f_2(y+5x)$

$$PI = \frac{1}{D^2 - DD' - 20D'^2}\left[e^{5x+y} + \sin(4x-y)\right]$$

$$= \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y} + \frac{1}{D^2 - DD' - 20D'^2} \sin(4x-y)$$

$= PI_1 + PI_2$

$$PI_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}$$

$D \to a = 5$
$D' \to b = 1$

$$= \frac{1}{25 - 5 - 20} e^{5x+y}$$

$$= \frac{x}{2D - D} e^{5x+y} = \frac{x}{2(5) - 1} e^{5x+y}$$

$$= \frac{x}{9} e^{5x+y} \qquad PI_1 = \frac{x}{9} e^{5x+y}$$

$$PI_2 = \frac{1}{D^2 - DD' - 20DD'^2} \sin(4x-y)$$

$a = 4 \quad b = -1$
$D^2 \to -a^2 = -16$
$DD' \to -ab = +4$
$D'^2 \to -b^2 = -(-1)^2 = -1$

$$= \frac{1}{-16 - 4 - 20(-1)} \sin(4x-y)$$

$$= x \frac{1}{2D - D'} \sin(4x-y)$$

$$= x \frac{2D + D'}{(2D)^2 - D'^2} \sin(4x-y) = \frac{x\left[2D\sin(4x-y) + D'(\sin(4x-y))\right]}{-64+1}$$

$$= \frac{-x}{63}\left[8\cos(4x-y) - \cos(4x-y)\right]$$

$$= \frac{-7x}{63}\cos(4x-y)$$

$$= \frac{-x}{9}\cos(4x-y)$$

The soln is $z = CF + PI$

$$= f_1(y-4x) + f_2(y+5x) + \frac{x}{9}e^{5x+y} - \frac{x}{9}\cos(4x-y)$$

HW:

1. $(D^2+4DD'-5D'^2)z = e^{2x-y} + \sin(x-2y)$
2. $(D^2-DD'-30D'^2)z = xy + e^{6x+y}$
3. $r+s-6t = y\cos x$.

Type. IV $\quad RHS = f(x,y) = e^{ax+by}x^m y^n$ (or) $e^{ax+by}\cos(ax+by)$, $\sin(ax+by)$

$$PI = \frac{1}{\phi(D,D')}e^{ax+by}x^m y^n$$

Replace $D \to D+a$; $D' \to D'+b$ then III rule or type II rule

1. Solve: $(D^2-2DD'+D'^2)z = x^2y^2e^{x+y}$

AE is $m^2-2m+1=0$

$(m-1)^2=0 \quad m=1,1$

$\therefore$ The roots are real and equal

CF is $z = f_1(y+mx) + x f_2(y+mx)$

$$= f_1(y+x) + x f_2(y+x)$$

$$PI = \frac{1}{D^2-2DD'+D'^2}x^2y^2e^{x+y}$$

Replace $D \to D+1$, $D' \to D'+1$

$$= \frac{1}{(D+1)^2 - 2(D+1)(D'+1) + (D'+1)^2} e^{x+y} x^2y^2$$

$$= \frac{1}{D^2 + 2D + 1 - 2DD' - 2D - 2D' - 2 + D'^2 + 2D' + 1} e^{x+y} x^2y^2$$

$$= \frac{1}{D^2 - 2DD' + D'^2} e^{x+y} \cdot x^2y^2$$

$$= e^{x+y} \frac{1}{D^2}\left[1 - \left(\frac{2D'}{D} - \frac{D'^2}{D^2}\right)\right]^{-1} x^2y^2$$

$$= e^{x+y} \frac{1}{D^2}\left[1 + \left(\frac{2D'}{D} - \frac{D'^2}{D^2}\right) + \dots\right] x^2y^2$$

$$= e^{x+y} \frac{1}{D^2}\left[x^2y^2 + \frac{2D'}{D} x^2y^2 - \frac{D'^2}{D^2} x^2y^2\right]$$

$$= e^{x+y}\left[\frac{1}{D^2} x^2y^2 + \frac{2D'}{D^3} x^2y^2 - \frac{D'^2}{D^4} x^2y^2\right]$$

$$= e^{x+y}\left[\frac{1}{D^2} x^2y^2 + \frac{4}{D^3} x^2y - \frac{2}{D^4} x^2\right]$$

$$= e^{x+y}\left[\frac{x^4y^2}{12} + \frac{4x^5y}{60} - \frac{2x^6}{360}\right]$$

$$= e^{x+y}\left[\frac{x^4y^2}{12} + \frac{x^5y}{15} - \frac{x^6}{180}\right]$$

$\therefore$ The soln is $z = CF + PI$

$$= f_1(y+x) + x f_2(y+x) + e^{x+y}\left[\frac{x^4y^2}{12} + \frac{x^5y}{15} - \frac{x^6}{180}\right]$$

Solve $r + s - 6t = y\cos x$

Given: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

$(D^2 + DD' - 6D'^2)z = y\cos x$

$m^2 + m - 6 = 0. \Rightarrow (m+3)(m-2) = 0.$

$m = -3, 2.$

$CF = f_1(y-3x) + f_2(y+2x)$

$PI = \frac{1}{(D^2 + DD' - 6D'^2)} y\cos x$

$= \frac{1}{(D+3D')(D-2D')} y\cos x$ factor $\rightarrow D - 2D'$ where $y = c - 2x$ $D \rightarrow c$ $D' \rightarrow x$

$= \frac{1}{(D+3D')}\int (c-2x)\cos x \, dx$

$= \frac{1}{(D+3D')}\left[(c-2x)\sin x - (-2)(-\cos x)\right]$

$= \frac{1}{(D+3D')}\left[y\sin x - 2\cos x\right]$ factor $\rightarrow D + 3D'$ $y \rightarrow c + 3x$

$= \int\left[(c+3x)\sin x - 2\cos x\right] dx$

$= (c+3x)(-\cos x) - 3(-\sin x) - 2\sin x$

$= -y\cos x + 3\sin x - 2\sin x$

$= -y\cos x + \sin x$