Linear PDE of 2nd order with constant coefficients

Homogeneous Linear PDE's:

A Linear PDE with constant coefficients in which all the partial derivatives are of the same order is called homogenous, otherwise it is called non-homogenous

Example:

Homogenous Equation:-

$$\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = \sin x.$$

Non Homogeneous Equation:

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial z}{\partial x} + 7\frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial y^2} = e^{x+y}$$

Notation: $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

Method of finding complementary function (CF):

Let the given equation be of the form

$$f(D, D')z = f(x,y)$$

Put $D = m$, $D' = 1$

$$f(m,1) = 0 \Rightarrow a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0$$

Let the roots of the eqn be $m_1, m_2, \dots m_n$

Roots	Complementary function
1. The Roots are different $m_1, m_2, \dots m_n$	$CF = f_1(y+m_1x) + f_2(y+m_2x) + \dots + f_n(y+m_nx)$
2. The roots are equal $m_1 = m_2 = \dots = m_n = m$	$CF = f_1(y+mx) + x f_2(y+mx) + \dots + x^{n-1} f_n(y+mx)$

General solution is $y = CF + PI$

RHS = 0 $(z = CF)$

1. Solve $(D^2 - 6DD' + 9D'^2)z = 0$

Put $D = m$, $D' = 1$

The auxillary equation is,

$$m^2 - 6m + 9 = 0$$
$$(m-3)(m-3) = 0$$
$$m = 3, 3 \text{ (equal)}$$

The solution is $z = CF$
$$= f_1(y+3x) + x f_2(y+3x)$$

Type-I

1. Solve $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

The auxillary equation is,

$$m^2 - 5m + 6 = 0$$
$$(m-3)(m-2) = 0$$
$$m = 2, 3$$

$$CF = f_1(y+2x) + f_2(y+3x)$$

$$PI = \frac{1}{D^2 - 5DD' + 6D'^2} e^{x+y}$$

Replace
$D \to a = 1$
$D' \to b = 1$

$$= \frac{1}{1-5+6} e^{x+y}$$
$$= \frac{1}{2} e^{x+y}$$

The solution is $z = CF + PI$
$$= f_1(y+2x) + f_2(y+3x) + \frac{e^{x+y}}{2}$$

3. Type-I Solve : $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

The auxillary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0.$$

$$m = 2, 2 \quad (\text{equal})$$

$$CF = f_1(y+2x) + x f_2(y+2x)$$

$$PI = \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$$

Replace
$D \to a = 2$
$D' \to b = 1$

$$= \frac{1}{2^2 - 4(2)(1) + 4(1)^2} e^{2x+y}$$

$$= \frac{1}{4-8+4} e^{2x+y}$$

$$= x \frac{1}{2D - 4D'} e^{2x+y} = x \frac{1}{2(2) - 4(1)} e^{2x+y}$$

$$= x^2 \frac{e^{2x+y}}{2} = \frac{x^2}{2} e^{2x+y}.$$

The solution is $z = CF + PI$

$$= f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$$

4. Solve : $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0.$

Given: $(2D^2 + 5DD' + 2D'^2)z = 0.$

A.E is $2m^2 + 5m + 2 = 0$

$2m^2 + 4m + m + 2 = 0$

$2m(m+2) + 1(m+2) = 0$

$(2m+1)(m+2) = 0 \quad m_1 = \frac{-1}{2}, \ m_2 = -2$
$\Rightarrow$ roots are different

CF $\Rightarrow$ $z = f_1(y - \frac{1}{2}x) + f_2(y - 2x)$

PI $\Rightarrow$ PI $= 0$.

$\therefore$ solution is $z = CF + PI$

$= f_1(y - \frac{1}{2}x) + f_2(y - 2x)$

To find Particular Integral (PI)

Type I: RHS $= f(x,y) = e^{ax+by}$

$$PI = \frac{1}{\phi(D,D')} e^{ax+by}$$

Replace $D \to a$, $D' \to b$

then $PI = \frac{1}{\phi(a,b)} e^{ax+by}$, provided $\phi(a,b) \neq 0$

If $\phi(a,b) = 0$ then differentiate the Denominator w.r.t 'D' and multiply by x in Numerator.

2. Solve: $(2D^2 - 2DD' + D'^2)z = 2e^{3y} + e^{x+y}$.

AE is $2m^2 - 2m + 1 = 0$

$$m = \frac{-(-2) \pm \sqrt{(2)^2 - 4(2)(1)}}{2(2)} = \frac{2 \pm \sqrt{4-8}}{4} = \frac{2 \pm 2i}{4} = \frac{1 \pm i}{2}$$

$m = \frac{1}{2} \pm \frac{1}{2}i$

$\therefore$ The roots are imaginary

CF is $z = f_1(y + x(\frac{1}{2} + \frac{1}{2}i)) + f_2(y + (\frac{1}{2} - \frac{1}{2}i)x)$

PI $\Rightarrow$ $PI_1 = \frac{1}{2D^2 - 2DD' + D'^2} 2e^{3y}$ $\rightarrow$ $D = 0$ & $D' = 3$

$$= \frac{1}{2(0) - 2(0)(3) + (3)^2} 2e^{3y} = \frac{2}{9}e^{3y}.$$

$$PI_2 = \frac{1}{2D^2 - 2DD' + D'^2} e^{x+y} \qquad D = 1, \; D' = 1$$

$$= \frac{1}{2(1)^2 - 2(1)(1) + (1)^2} e^{x+y}$$

$$= e^{x+y}$$

$\therefore$ The solution is $z = CF + PI$

$$z = f_1\left(y + \left(\tfrac{1}{2} + \tfrac{1}{2}i\right)x\right) + f_2\left(y + \left(\tfrac{1}{2} - \tfrac{1}{2}i\right)x\right) + \frac{2}{9} e^{3y} + e^{x+y}$$

4. Solve : $(D^2 - 3DD' + 2D'^2)z = e^{3x+2y}$

AE is $m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0.$

$m_1 = 1, \; m_2 = 2$

CF is $z = f_1(y+x) + f_2(y+2x)$

$$PI = \frac{1}{D^2 - 3DD' + 2D'^2} e^{3x+2y} \qquad D = 3, \; D' = 2$$

$$= \frac{1}{(3)^2 - 3(3)(2) + 2(2)^2} e^{3x+2y}$$

$$= \frac{1}{9 - 18 + 8} e^{3x+2y} = -e^{3x+2y}$$

$\therefore$ The solution is $z = CF + PI$

$$= f_1(y+x) + f_2(y+2x) - e^{3x+2y}.$$

5. Solve: $(D^2 - DD' - 20D'^2)z = e^{5x+y}$

AE is $m^2 - m - 20 = 0$

$(m-5)(m+4) = 0 \qquad m_1 = -4, \; m_2 = 5$

$\therefore$ CF is $z = f_1(y - 4x) + f_2(y + 5x)$

5. $PI = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}$ $\quad D = 5, D' = 1$

$= \frac{1}{(5)^2 - (5)(1) - 20(1)} e^{5x+y} = \frac{1}{25 - 5 - 20} e^{5x+y}$

$= \frac{x}{2D - D'} e^{5x+y} = \frac{x}{2(5) - 1} e^{5x+y}$

$= \frac{x}{9} e^{5x+y}$

$\therefore$ The solution is $z = CF + PI$

$= f_1(y - 4x) + f_2(y + 5x) + \frac{x e^{5x+y}}{9}$

6. Solve: $(D^2 + 2DD' + D'^2)z = e^{x-y}$

AE is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)(m+1) = 0$.

$m = -1, -1$ Roots are ~~same~~ equal

$\therefore CF$ is $z = f_1(y - x) + x f_2(y - x)$

$PI = \frac{1}{D^2 + 2DD' + D'^2} e^{x-y}$ $\quad D = 1, D' = -1$

$= \frac{1}{(1)^2 + 2(1)(-1) + (-1)^2} e^{x-y} = \frac{e^{x-y}}{1 - 2 + 1}$

$= \frac{x}{2D + 2D'} e^{x-y} = \frac{x}{2(1) + 2(-1)} e^{x-y}$

$= \frac{x^2}{2} e^{x-y}$

The solution is $z = CF + PI$

$= f_1(y - x) + x f_2(y - x) + \frac{x^2}{2} e^{x-y}$.