Type-IV $f_1(x,p) = f_2(y,q)$

for this type, there is no Singular Integral.

1. Solve $q^2 - p = y - x$

Given : $q^2 - p = y - x$ (constant)

$q^2 - y = p - x = k$

Now, $q^2 - y = k$

$q^2 = k + y$

$q = \sqrt{k+y}$

$p - x = k$

$p = k + x$

WKT, $z = \int p\,dx + \int q\,dy$

$z = \int (k+x)\,dx + \int \sqrt{k+y}\,dy$

$= kx + \frac{x^2}{2} + \frac{(k+y)^{3/2}}{3/2} + c$

$= kx + \frac{x^2}{2} + \frac{2}{3}(k+y)^{3/2} + c$, which is the

Complete Integral.

2. Solve: $\sqrt{p} + \sqrt{q} = x + y$

Given:- $\sqrt{p} + \sqrt{q} = x + y$

$\sqrt{p} - x = -\sqrt{q} + y = k$

$\sqrt{p} - x = k$

$\sqrt{p} = k + x$

$p = (k+x)^2$

$y - \sqrt{q} = k$

$\sqrt{q} = y - k$

$q = (y-k)^2$

WKT, $z = \int p\,dx + \int q\,dy$

$z = \int (k+x)^2\,dx + \int (y-k)^2\,dy$

$z = \frac{(k+x)^3}{3} + \frac{(y-k)^3}{3} + c$, which is the complete Integral

3. Find the complete Integral of $xp - yq = y^2 - x^2$

Given! $xp - yq = y^2 - x^2$

$xp + x^2 = y^2 + yq = k$

$xp + x^2 = k$

$xp = k - x^2$

$p = \frac{k - x^2}{x}$

$p = \frac{k}{x} - x$

$y^2 + yq = k$

$yq = k - y^2$

$q = \frac{k - y^2}{y}$

$q = \frac{k}{y} - y$

wkt, $z = \int p\,dx + \int q\,dy$

$$= \int\left(\frac{k}{x} - x\right)dx + \int\left(\frac{k}{y} - y\right)dy$$

$$= k\log x - \frac{x^2}{2} + k\log y - \frac{y^2}{2} + c$$

$z = k\log(xy) - \left(\frac{x^2+y^2}{2}\right) + c$, which is the complete Integral.

Type 2:

3) Solve: $z = px + qy + \sqrt{1+p^2+q^2}$

Given: $z = px + qy + \sqrt{1+p^2+q^2}$

Complete Integral:

$$z = ax + by + \sqrt{1+a^2+b^2} \rightarrow (A)$$

Singular Integral:

$\frac{\partial z}{\partial a} = 0$

$$x + \frac{1(2a)}{2\sqrt{1+a^2+b^2}} = 0$$

$$x = \frac{-a}{\sqrt{1+a^2+b^2}} \rightarrow (1)$$

$\frac{\partial z}{\partial b} = 0$

$$y + \frac{2b}{2\sqrt{1+a^2+b^2}} = 0$$

$$y = \frac{-b}{\sqrt{1+a^2+b^2}} \rightarrow (2)$$

Squaring on both sides,

$$x^2 = \frac{a^2}{1+a^2+b^2}, \quad y^2 = \frac{b^2}{1+a^2+b^2}$$

Now,
$$x^2+y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-(x^2+y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-x^2-y^2 = \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2}$$

$$1-x^2-y^2 = \frac{1}{1+a^2+b^2}$$

Taking square root,

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{1+a^2+b^2}}$$

$$\Rightarrow \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

$$(1) \Rightarrow \quad x = -a\sqrt{1-x^2-y^2} \quad \Rightarrow a = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$(2) \Rightarrow \quad y = -b\sqrt{1-x^2-y^2} \quad \Rightarrow b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$(A) \Rightarrow \quad z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2.$$

HW 1. $z = px + qy + (pq)^{3/2}$

2. $z = px + qy + p^2q^2$