Solution of Standard types of first order PDE

A partial differential equation in which the partial derivative occurring are of the first degree is said to be linear, otherwise it is said to be non-linear.

Standard types:

Type 1 : $F(p, q) = 0$

Type 2 : $z = px + qy + f(p, q)$ [Clairaut's form]

Type 3 : $f(z, p, q) = 0$

Type 4 : $f_1(x, p) = f_2(y, q)$

Type 1 : Working Rule :

1. Let $z = ax + by + c$ be the complete integral. $p = \frac{\partial z}{\partial x} = a, q = \frac{\partial z}{\partial y} = b$
2. put $b = \phi(a)$ for general solution
3. There is no singular integral

1. Solve : $p + q = pq$

Soln: $p + q = pq \rightarrow (1)$

Let $z = ax + by + c \rightarrow (2)$

Complete Integral :

Diff partially w.r.t 'x' and y

$\frac{\partial z}{\partial x} = a$, $p = a$ | $\frac{\partial z}{\partial y} = b$, $q = b$

Sub the above values in (1) we get

$a + b = ab$

$a = ab - b$ $a = b(a-1) = b = \frac{a}{a-1}$

The complete Integral is,

$z = ax + \left(\frac{a}{a-1}\right)y + c \rightarrow (3)$

Singular Integral:

diff (3) p w.r.t 'a' and 'c' and equal to zero

$\frac{\partial z}{\partial a} = x + \left[\frac{(a-1)(1) - a(1)}{(a-1)^2}\right]y = 0$, $\frac{\partial z}{\partial c} = 1 \neq 0.$

There is no singular Integral

General Integral: put $c = \phi(a)$ in ③

$$z = ax + \left(\frac{a}{a-1}\right)y + \phi(a) \rightarrow ④$$

Diff ④ p wrt 'a',

$$\frac{\partial z}{\partial a} = x + \left(\frac{(a-1)(1) - a(1)}{(a-1)^2}\right)y + \phi'(a) = 0 \rightarrow ⑤$$

Eliminate 'a' b/w ④ & ⑤ we get the general solution

2. Solve $\sqrt{p} + \sqrt{q} = 1$

Soln: $\sqrt{p} + \sqrt{q} = 1 \rightarrow ①$ Let $z = ax + by + c$

Complete Integral:

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b$$

Sub the above values in ① we get,

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a} \qquad b = (1-\sqrt{a})^2$$

The complete Integral is

$$z = ax + (1-\sqrt{a})^2 y + c \rightarrow ②$$

Singular Integral:

$$\frac{\partial z}{\partial a} = x + 2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right)y = 0 \text{ , } \frac{\partial z}{\partial c} = 1 \neq 0$$

There is no singular Integral

General Integral:

put $c = \phi(a)$ in ②

$$z = ax + (1-\sqrt{a})^2 y + \phi(a) \rightarrow ③$$

Diff 3 p wrt 'a'

$$\frac{\partial z}{\partial a} = x + 2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right)y + \phi'(a) = 0 \rightarrow ④$$

Eliminate 'a' b/w ③ & ④ we get the

... Integral.

...) $p - q = 0$, 2) $p^2+q^2 - 4pq = 0$ 3) $p^2+q^2 = 4$