Method of Multipliers

$$Pp + Qq = R$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

I $\quad \dfrac{dx \pm dy \pm dz}{P \pm Q \pm R} = 0$

ii $\quad \dfrac{x\,dx \pm y\,dy \pm z\,dz}{xP \pm yQ \pm zR} = 0$

iii $\quad \dfrac{x^2 dx \pm y^2 dy \pm z^2 dz}{x^2 P \pm y^2 Q \pm z^2 R} = 0$

iv $\quad \dfrac{\frac{1}{x} dx \pm \frac{1}{y} dy \pm \frac{1}{z} dz}{\frac{1}{x} P \pm \frac{1}{y} Q \pm \frac{1}{z} R} = 0$

v $\quad \dfrac{\frac{1}{x^2} dx \pm \frac{1}{y^2} dy \pm \frac{1}{z^2} dz}{\frac{1}{x^2} P \pm \frac{1}{y^2} Q \pm \frac{1}{z^2} R} = 0$

vi $\quad \dfrac{l\,dx \pm m\,dy \pm n\,dz}{lP \pm mQ \pm nR} = 0$

1. Solve $x(y-z)p + y(z-x)q = z(x-y)$

$$Pp + Qq = R$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

I $\Rightarrow$ $$\frac{dx+dy+dz}{xy-xz+yz-xy+zx-zy}=0$$

$$dx+dy+dz=0$$

$$\int dx+\int dy+\int dz=0$$

$$x+y+z=c_1 \rightarrow (1)$$

iv $\Rightarrow$ $$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{x}\cdot x(y-z)+\frac{1}{y}y(z-x)+\frac{1}{z}z(x-y)}=0$$

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y-z+z-x+x-y}=0$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0.$$

$$\int\frac{1}{x}dx+\int\frac{1}{y}dy+\int\frac{1}{z}dz=0$$

$$\log x+\log y+\log z=\log c_2$$

$$\log(xyz)=\log c_2$$

$$xyz=c_2$$

$$\phi(c_1,c_2)=0$$

$$\phi(x+y+z,\ xyz)=0.$$

2. Solve $x(z^2-y^2)p+y(x^2-z^2)q=z(y^2-x^2)$

$$Pp+Qq=R$$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{x(z^2-y^2)}=\frac{dy}{y(x^2-z^2)}=\frac{dz}{z(y^2-x^2)}$$

II $\Rightarrow$ $$\frac{x\,dx + y\,dy + z\,dz}{x^2(z^2-y^2) + y^2(x^2-z^2) + z^2(y^2-x^2)} = 0$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^2z^2 - x^2y^2 + y^2x^2 - y^2z^2 + y^2z^2 - x^2z^2} = 0$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1 \rightarrow (1)$$

IV $\Rightarrow$ $$\frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{\frac{1}{x}\,x(z^2-y^2) + \frac{1}{y}\cdot y(x^2-z^2) + \frac{1}{z}\cdot z(y^2-x^2)} = 0$$

$$\frac{\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz}{z^2 - y^2 + x^2 - z^2 + y^2 - x^2} = 0$$

$$\frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$$

$$\int \frac{1}{x}\,dx + \int \frac{1}{y}\,dy + \int \frac{1}{z}\,dz = 0.$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2 \rightarrow (2)$$

By combining (1) & (2)

$$\phi(c_1, c_2) = 0$$

$$\phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2},\ xyz\right) = 0.$$

3. Solve $(mz-ny)p+(nx-lz)q = ly-mx$

$$Pp+Qq=R$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

II $\Rightarrow$ $$\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = 0$$

$$\frac{xdx+ydy+zdz}{xmz-xny+ynx-lyz+lyz-mxz} = 0$$

$$xdx+ydy+zdz=0$$

$$\int xdx+\int ydy+\int zdz=0.$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = c_1 \rightarrow (1)$$

IV $\Rightarrow$ $$\frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)} = 0$$

$$\frac{ldx+mdy+ndz}{lmz-lny+mnx-lmz+lny-mnx} = 0$$

$$ldx+mdy+ndz=0$$

$$\int ldx+\int mdy+\int ndz=0$$

$$lx+my+nz = c_2 \rightarrow (2).$$

Combining (1) & (2),

$$\phi(c_1, c_2) = 0$$

$$\phi\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2},\ lx+my+nz\right)=0.$$

(4) Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$$Pp + Qq = R$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

(i) $\Rightarrow$

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\frac{1}{x^2}\cdot x^2(y-z) + \frac{1}{y^2}\cdot y^2(z-x) + \frac{1}{z^2}\cdot z^2(x-y)} = 0.$$

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y-z+z-x+x-y} = 0.$$

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

$$\int\frac{1}{x^2}dx + \int\frac{1}{y^2}dy + \int\frac{1}{z^2}dz = 0.$$

$$\left(\frac{-1}{x}\right)+\left(\frac{-1}{y}\right)+\left(\frac{-1}{z}\right) = c_1$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = c_1$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}\cdot x^2(y-z) + \frac{1}{y}\cdot y^2(z-x) + \frac{1}{z}\cdot z^2(x-y)} = 0$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(y-z) + y(z-x) + z(x-y)} = 0.$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{xy - xz + yz - xy + xz - yz} = 0.$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0.$$

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0.$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2. \rightarrow (2)$$

By combining (1) & (2)

$$\phi(c_1, c_2) = 0$$

$$\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0.$$

(5) Solve: $z(x-y) = x^2p - y^2q$

$Pp + Qq = R$ $\qquad P = x^2 \quad Q = -y^2 \quad R = z(x-y)$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)} = \frac{dx+dy}{(x+y)(x-y)}$$

Taking first two members

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

Integrating $\int \frac{1}{x^2}dx = -\int \frac{1}{y^2}dy$

$$\frac{-1}{x} = \frac{1}{y} - c_1$$

$$c_1 = \frac{1}{x} + \frac{1}{y}$$

Taking last two members

$$\frac{dz}{z(x-y)} = \frac{dx+dy}{(x+y)(x-y)}$$

$$\frac{dz}{z} = \frac{d(x+y)}{(x+y)}$$

Integrating, $\int \frac{dz}{z} = \int \frac{d(x+y)}{(x+y)}$

$$\log z = \log(x+y) + \log c_2.$$

$$\log z - \log(x+y) = \log c_2$$

$$\log\left(\frac{z}{x+y}\right) = \log c_2$$

$$c_2 = \frac{z}{x+y}$$

$\therefore \quad \phi(c_1, c_2) = 0$

$$\phi\left(\frac{1}{x} + \frac{1}{y}, \frac{z}{x+y}\right) = 0.$$

⑥ Solve $(y^2+z^2)p - xyq + xz = 0.$

$Pp + Qq = R \qquad P = y^2+z^2, \; Q = -xy \quad Z = -xz$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$$

I $\quad \dfrac{xdx + ydy + zdz}{x(y^2+z^2) - y(xy) - z(xz)} = 0$

$$\frac{xdx + ydy + zdz}{xy^2 + xz^2 - xy^2 - xz^2} = 0$$

$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0.$

$\int x\,dx + \int y\,dy + \int z\,dz = 0$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$

$x^2 + y^2 + z^2 = c_1$

$\Rightarrow \quad \frac{dy}{-xy} = \frac{dz}{-xz}$

$\frac{dy}{y} = \frac{dz}{z}$

Integrating, $\int \frac{dy}{y} = \int \frac{dz}{z}$

$\log y = \log z + \log c_2$

$\log y - \log z = \log c_2$

$\log\left(\frac{y}{z}\right) = \log c_2$

$c_2 = \frac{y}{z}$

$\Rightarrow \phi(c_1, c_2) = 0$

$\phi\left(x^2+y^2+z^2, \frac{y}{z}\right) = 0.$

7) Solve $(x+2z)p + (2xz - y)q = x^2+y.$

$P = x+2z \quad Q = 2xz - y \quad R = x^2+y$

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{x+2z} = \frac{dy}{2xz-y} = \frac{dz}{x^2+y}$ [Multipliers : $y, x, -2z$ or $2x, -2z, -2z$]

$$\text{I} \Rightarrow \quad \frac{y\,dx + x\,dy - 2z\,dz}{y(x+2z) + x(2xz - y) - 2z(x^2+y)}$$

$$\frac{y\,dx + x\,dy - 2z\,dz}{xy + 2yz + 2x^2z - xy - 2x^2z - 2yz}$$

$$y\,dx + x\,dy - 2z\,dz = 0.$$

$$d(xy - z^2) = 0$$

Integrating $\int d(xy - z^2) = 0.$

$$xy - z^2 = c_1$$

$$\text{II} \Rightarrow \quad \frac{2x\,dx - 2dy - 2dz}{2x(x+2z) - 2(2xz - y) - 2(x^2+y)}$$

$$\frac{2x\,dx - 2dy - 2dz}{2x^2 + 4xz - 4xz + 2y - 2x^2 - 2y}$$

$$2x\,dx - 2dy - 2dz = 0.$$

$$d\left(\frac{2x^2}{2} - 2y - 2z\right) = 0$$

$$\int d(x^2 - 2y - 2z) = 0$$

$$x^2 - 2y - 2z = c_2$$

$$\phi(c_1, c_2) = 0$$

$$\phi(xy - z^2,\ x^2 - 2y - 2z) = 0.$$