Lagrange's Linear Equations

The equations of the form $Pp + Qq = R$ is known as Lagrange's Linear equation, where P, Q are functions of x, y, z

To solve this equation, it is enough to solve the subsidary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The auxillary eqn can be solved in 2 ways

1. Method of grouping
2. Method of Multipliers

Method of Grouping:-

$$Pp + Qq = R$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\left.\begin{array}{ll} \text{I} & \int \frac{dx}{P} = \int \frac{dy}{Q} \\ \text{II} & \int \frac{dy}{Q} = \int \frac{dz}{R} \\ \text{III} & \int \frac{dz}{R} = \int \frac{dx}{P} \end{array}\right\} \Rightarrow \phi(c_1, c_2) = 0.$$

1. Solve $x^2p + y^2q = z^2$

$$Pp + Qq = R$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}.$$

$$\Rightarrow \int \frac{dx}{P} = \int \frac{dy}{Q}$$

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$\int x^{-2} dx = \int y^{-2} dy$$

$$-x^{-1} + c = -y^{-1} + c$$

$$\frac{-1}{x} + \frac{1}{y} = c_1 \longrightarrow (1)$$

$$\Rightarrow \int \frac{dy}{Q} = \int \frac{dz}{R}$$

$$\int \frac{dy}{y^2} = \int \frac{dz}{z^2}$$

$$\int y^{-2} dy = \int z^{-2} dz$$

$$\frac{y^{-1}}{-1} + c = \frac{z^{-1}}{-1} + c$$

$$\frac{-1}{y} + \frac{1}{z} = c_2 \longrightarrow (2)$$

Combining eqns (1) & (2)

$$\phi(c_1, c_2) = 0.$$

$$\phi\left(\frac{-1}{x} + \frac{1}{y}, \frac{-1}{y} + \frac{1}{z}\right) = 0.$$

2) $\frac{y^2z}{x}p + xzq = y^2$

$Pp+Qq=R$

$\frac{dx}{P} = \frac{dy}{Q}$

$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}.$

I $\Rightarrow \int \frac{dx}{\frac{y^2z}{x}} = \int \frac{dy}{xz}$

$\int \frac{x\,dx}{y^2z} = \int \frac{dy}{xz}$

$\int x^2dx = \int y^2dy$

$\frac{x^3}{3}+c = \frac{y^3}{3}+c$

$\Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = c_1 \rightarrow (1)$

III $\Rightarrow \int \frac{dz}{y^2} = \int \frac{dx}{\frac{y^2z}{x}}$

$\int \frac{dz}{y^2} = \int \frac{x\,dx}{y^2z}$

$\int z\,dz = \int x\,dx$

$\frac{z^2}{2}+c = \frac{x^2}{2}+c$

$\frac{z^2}{2} - \frac{x^2}{2} = c_2 \rightarrow (2)$

$\phi\left(\frac{x^3}{3} - \frac{y^3}{3}, \frac{z^2}{2} - \frac{x^2}{2}\right) = 0.$