3. Partial Differential Equations

Definition:-

A partial differential equation is an equation, involving a function of 2 or more variable and sum of its partial derivatives.

Notations:-

$$P = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

Formation of Partial differential Equations:

i) Eliminating Arbitrary constants

ii) Eliminating Arbitrary function.

Eliminating Arbitrary constant:

Type 1: Number of Arbitrary constant $\leq$ Number of Independent Variable, then we get the 1st order partial differential equation.

1. Form the PDE by eliminating Arbitrary constant form $z = ax + by + a^2 + ab + b^2$.

Diff partially w.r.t x

$$\frac{\partial z}{\partial x} = a$$

$$\boxed{P = a}$$

A.C	IV
a, b	x, y
2	2
No of A.C = 2	
No of IV = 2	

Diff P w.r.t y.

$$\frac{\partial z}{\partial y} = b \rightarrow \boxed{q = b}.$$

$$z = ax + by + a^2 + ab + b^2.$$

$$= px + qy + p^2 + pq + q^2.$$

2. Form the P.d.e by E.A.C from

$$z = (x-a)^2 + (y-b)^2 + 1$$

Diff P w.r.to 'x'.

$$\frac{\partial z}{\partial x} = 2(x-a)(1)$$

$$p = 2(x-a) \Rightarrow \boxed{\frac{p}{2} = (x-a)}$$

Diff P w.r.t 'y'

$$\frac{\partial z}{\partial y} = 2(y-b)(1)$$

$$\boxed{\frac{q}{2} = y-b}$$

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 + 1.$$

Form the P.d.e by eliminating Arbitrary Constant from

$$\log(az-1) = x + ay + b$$

Diff P w.r.to x,

$$\frac{1}{(az-1)} \frac{\partial}{\partial x}(az-1) = 1$$

$$\frac{1}{az-1}\, a\,\frac{\partial z}{\partial x} = 1$$

$$\frac{ap}{az-1} = 1 \longrightarrow (1).$$

Diff P wrt y,

$$\frac{1}{(az-1)}\,\frac{\partial}{\partial y}(az-1) = a$$

$$\frac{1}{(az-1)}\, a\,\frac{\partial z}{\partial y} = a$$

$$\frac{aq}{az-1} = a$$

$$\frac{q}{az-1} = 1 \longrightarrow (2).$$

Equating (1) & (2).

$$\frac{ap}{az-1} = \frac{q}{az-1} \Rightarrow \boxed{a = \frac{q}{p}} \rightarrow (3).$$

$$q = az-1$$

$$q+1 = az$$

$$\boxed{a = \frac{q+1}{z}} \quad (\text{from } (2)) \rightarrow (4)$$

$z = px + qy + \sqrt{p^2+q^2}$

4) $ax + by + \sqrt{a^2+b^2}$

5) $(x^2+a^2)(y^2+b^2) \Rightarrow \frac{p^2}{4} + \frac{q^2}{4} + 1$

6) $\cancel{(x^2-a^2)}$ $(x-a)^2 + (y-b)^2 + z^2 = 1$

$(p^2+q^2+1)z^2 = 1$

Equating (3) & (4)

$$\frac{q+1}{z} = \frac{q}{p}$$

$$(q+1)p = qz \Rightarrow pq + p = qz$$

$$zq - pq - p = 0.$$

Type 2:

~~No~~ Number of A.C > No of independent variable then we get the 2^{nd} order Pde.

1. Form the Pde by EAC from $z = ax + by + cxy$.

Diff P ~~w~~.r.to x

$$\frac{\partial z}{\partial x} = a + cy$$

$$\boxed{p - cy = a}$$

Diff p w.r.t y

$$\frac{\partial z}{\partial y} = b + cx$$

$$q - cx = b$$

Diff partially w.r.t x

$$\frac{\partial z}{\partial x} = a + cy$$

$$\frac{\partial^2 z}{\partial x^2} = 0 . \Rightarrow \boxed{r = 0}$$

Diff p w.r.t y

$$\frac{\partial z}{\partial y} = b + cx \Rightarrow \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow \boxed{t = 0}$$

$$\Rightarrow \frac{\partial z}{\partial x} = a + cy$$

Diff p w.r.to y $\quad \frac{\partial^2 z}{\partial x \partial y} = c \Rightarrow \boxed{s = c}$

$$z = (p - cy)x + (q - cx)y + sxy$$

$$= px - cyx + qy - cxy + sxy$$

$$= px + qy - \cancel{2}cxy + sxy$$

Sub $\boxed{c = s}$

$$z = px + qy - sxy.$$

Eliminating Arbitrary Functions

Type 1: Eliminating one arbitrary function then we get 1st order partial differential equation.

1. Form the p.d.e by eliminating arbitrary function from

$$z = f(x^2+y^2+z^2)$$

Diff p w.r.to 'x'

$$\frac{\partial z}{\partial x} = f'(x^2+y^2+z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right)$$

$$p = f'(x^2+y^2+z^2)\,2(x + zp) \rightarrow (1).$$

Diff p w.r.to 'y'

$$\frac{\partial z}{\partial y} = f'(x^2+y^2+z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right)$$

$$q = 2f'(x^2+y^2+z^2)(y + zq) \rightarrow (2)$$

Divide eqn (1) & (2)

$$\frac{p}{q} = \frac{f'(x^2+y^2+z^2)\,2(x+zp)}{f'(x^2+y^2+z^2)\,2(y+zq)} = \frac{x+zp}{y+zq}$$

$$p(y+zq) = q(x+zp)$$
$$py + zpq = qx + zpq$$
$$py - qx = 0 \Rightarrow \boxed{py = qx}$$

2. $z = xyz + \phi(x^2+y^2-z^2)$

Diff p w.r.to 'x'.

$$\frac{\partial z}{\partial x} = \left[z + x\frac{\partial z}{\partial x}\right]y + \phi'(x^2+y^2-z^2)\left(2x - 2z\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial z}{\partial x} = zy + xy\frac{\partial z}{\partial x} + \phi'(x^2+y^2-z^2)\left(2x - 2z\frac{\partial z}{\partial x}\right)$$

$$p - zy - xyp = 2\phi'(x^2+y^2-z^2)(x - zp) \rightarrow (1)$$

Diff p w.r.to 'y'

$$\frac{\partial z}{\partial y} = \left(z + y\frac{\partial z}{\partial y}\right)x + \phi'(x^2+y^2-z^2)\left(2y - 2z\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial z}{\partial y} = zx + xy\frac{\partial z}{\partial y} + \phi'(x^2+y^2-z^2).2\left(y - z\frac{\partial z}{\partial y}\right)$$

$$q - zx - xyq = 2\phi'(x^2+y^2-z^2)(y - zq) \rightarrow (2).$$

$\div$ (1) & (2)

$$\frac{p - zy - xyp}{q - zx - xyq} = \frac{\phi'(x^2+y^2-z^2)\,2\,(x-zp)}{\phi'(x^2+y^2-z^2)\,2(y-zq)}$$

$$\frac{p - zy - xyp}{q - zx - xyq} = \frac{x - zp}{y - zq}$$

$$(p - zy - xyp)(y - zq) = (x - zp)(q - zx - xyq)$$

$$py - zy^2 - xy^2p - zpq + z^2yq + xyzpq = xq - zx^2 - x^2yq - zpq + z^2xp + xyzpq$$

$$py - zy^2 - xy^2p + z^2yq - xq + zx^2 + x^2yq \mp z^2xp = 0$$

Type 2:

Eliminating two arbitrary function then we get second order partial Differential Equation.

1. Eliminating Arbitrary function from $z = f(ax+by) + g(cx+dy)$

Diff p w.r.to 'x'.

$$\frac{\partial z}{\partial x} = f'(ax+by)a + g'(cx+dy)c \rightarrow ①$$

$$p = a f'(ax+by) + c g'(cx+dy) \rightarrow ①$$

Diff p w.r.to 'y'.

$$\frac{\partial z}{\partial y} = f'(ax+by)b + g'(cx+dy)d$$

$$q = b f'(ax+by) + d g'(cx+dy) \rightarrow ②$$

Diff ① p w.r.to 'x'

$$\frac{\partial^2 z}{\partial x^2} = af''(ax+by)a + cg''(cx+dy).c$$

$$r = a^2 f''(ax+by) + c^2 g''(cx+dy) \rightarrow ③$$

Diff ② p w.r.to 'y'

$$\frac{\partial^2 z}{\partial y^2} = bf''(ax+by).b + d g''(cx+dy) d$$

$$t = b^2 f''(ax+by) + d^2 g''(cx+dy) \rightarrow ④$$

Diff ① p.w.r.to 'y'

$$\frac{\partial^2 z}{\partial x \partial y} = f''(ax+by)ab + g''(cx+dy)cd$$

$$s = abf''(ax+by) + cd\, g''(cx+dy) \rightarrow ⑤$$

$$\begin{vmatrix} r & a^2 & c^2 \\ s & ab & cd \\ t & b^2 & d^2 \end{vmatrix} = 0$$

$r(\cancel{abd^2 - b^2cd}) - a^2(sd^2 - tcd) + c^2(sb^2 - abt) = 0$

$r(abd^2 - b^2cd)$

$rabd^2 - rb^2cd - a^2sd^2 + a^2tcd + sc^2b^2 - abc^2t = 0.$

Type 3: Elimination of arbitrary function from $f(u,v) = 0$.

$u, v \rightarrow$ functions of x, y.

$f(u,v) = 0$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

1. Form the PDE by Eliminating Arbitrary function from $\phi(x^2+y^2+z^2, ax+by+cz) = 0$.

$u = x^2+y^2+z^2 \qquad v = ax+by+cz$

$\frac{\partial u}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}$

$= 2x + 2zp$

$= 2(x+zp)$

$\frac{\partial u}{\partial y} = 2y + 2z\frac{\partial z}{\partial y}$

$= 2y + 2zq$

$= 2(y+zq)$

$\frac{\partial v}{\partial x} = a + c\frac{\partial z}{\partial x}$

$= a + cp$

$\frac{\partial v}{\partial y} = b + c\frac{\partial z}{\partial y}$

$= b + cq$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

$$\begin{vmatrix} 2(x+zp) & a+cp \\ 2(y+zq) & b+cq \end{vmatrix} = 0.$$

$$2(x+zp)(b+cq) - (a+cp)(2(y+zq)) = 0.$$

$$2[xb + cxq + zbp + zc\cancel{pq} - ay - azq - cpy - zc\cancel{pq}] = 0$$

$$2[xb + cxq + zbp - ay - azq - cpy] = 0.$$