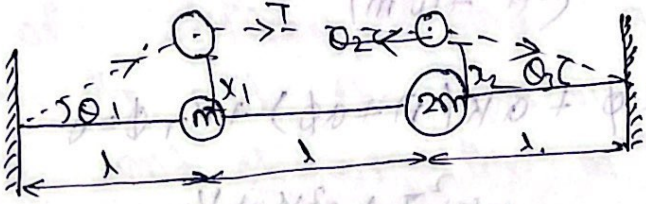
Determine the two natural frequencies and mode shapes for the system shown in Fig. 5.40. The string is stretched with a large Tension T.



Assume the Tension in the string as T and it does not change for small values of oscillation.

$$\sin\theta_1 = \frac{x_1}{\lambda} \qquad \sin\theta_2 = \frac{(x_2 - x_1)}{\lambda}$$

$$\sin\theta_3 = \frac{x_2}{\lambda_1}$$

equation of motion for left ball

$$m\ddot{x}_1 = -T\sin\theta_1 - T\sin\theta_2$$

$$m\ddot{x}_1 = -T\frac{x_1}{\lambda} + T\frac{(x_2 - x_1)}{\lambda}$$

$$m\ddot{x}_1 + \frac{2Tx_1}{\lambda} - \frac{Tx_2}{\lambda} = 0.$$

equation of motion for right ball mass

$$2m\ddot{x}_2 = +T\sin\theta_2 - T\sin\theta_3.$$

$$2m\ddot{x}_2 + \frac{2Tx_2}{l} - \frac{Tx_1}{l} = 0.$$

Assuming

$$x_1 = A_1 \sin\omega t$$

$$x_2 = A_2 \sin\omega t$$

$$\left(-m\omega^2 + \frac{2T}{\lambda}\right)A_1 - \frac{T}{\lambda}A_2 = 0$$

$$\left(-2m\omega^2 + \frac{2T}{\lambda}\right)A_2 - \frac{T}{\lambda}A_1 = 0$$

$$\frac{A_1}{A_2} = \frac{(T/l)}{-m\omega^2 + \frac{2T}{l}} = \frac{-2m\omega^2 + \frac{2T}{l}}{(T/l)}$$

$$\left(-m\omega^2 + \frac{2T}{l}\right)\left(-2m\omega^2 + \frac{2T}{l}\right) - \frac{T^2}{l^2} = 0$$

$$\omega^4 - \omega^2 \frac{3T}{ml} + \frac{3}{2}\frac{T^2}{m^2 l^2} = 0$$

$$\omega^2 = \frac{T}{ml}\,\frac{3\pm\sqrt{3}}{2}.$$

$$\left(\frac{A_1}{A_2}\right)_{\omega_1} = \frac{T/l}{-m\omega^2 + \frac{2T}{l}} = \frac{\left(\frac{T}{l}\right)}{-\frac{mT}{ml}\left(\frac{3\pm\sqrt{3}}{2}\right) + \frac{2T}{l}}$$

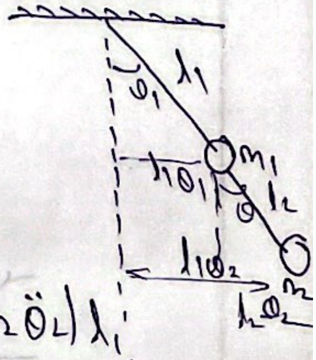
$$= \frac{2}{1+\sqrt{3}}$$

$$= \frac{2(1-\sqrt{3})}{(1-3)}$$

$$= -1+\sqrt{3}.$$

$$\therefore \left(\frac{A_1}{A_2}\right)_{\omega_2} = -1-\sqrt{3}.$$

4 Determine the natural frequency of oscillation of the double pendulum as shown in fig.



$m_1 = m_2 = 5\,kg$ $\therefore m_1 = m_2 = m = 5\,kg$

$l_1 = l_2 = 25\,cm$ $l_1 = l_2 = l = 25\,cm.$

$$I_1\ddot{\theta}_1 = -m_1 g l_1\theta_1 - m_2 g l_1\theta_1 - m_2(l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2)l_1$$

$$m_1 l_1^2\ddot{\theta}_1 = -m_1 g l_1\theta_1 - m_2 g l_2\theta_1 - m_2(l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2)l_1$$

$$m_2 l_2^2\ddot{\theta}_2 = -m_2 g l_2\theta_2 - m_2 l_1 l_2\ddot{\theta}_1$$

$$\ddot{\theta}_2 l_2^2 + g l_2 \theta_2 + l_1 l_2 \ddot{\theta}_1 = 0$$

$$\ddot{\theta}_2 + \frac{l_1 l_2}{l_2^2}\ddot{\theta}_1 + \frac{g l_2 \theta_2}{l_2^2} = 0$$

$$\ddot{\theta}_2 + \frac{l_1}{l_2}\ddot{\theta}_1 + \frac{g}{l_2}\theta_2 = 0.$$

$$\ddot{\theta}_1 + \frac{m_2 l_2}{(m_1+m_2) l_1}\ddot{\theta}_2 + \frac{g}{l_1}\theta_1 = 0$$

Let us assume the solution of the form.

$$\theta_1 = A_1 \sin\omega t \quad , \theta_2 = A_2 \sin\omega t$$

$$\ddot{\theta}_1 = -\omega^2 A_1 \sin\omega t \quad \ddot{\theta}_2 = -\omega^2 A \sin\omega t.$$

$$-\omega^2 A_2 + \frac{l_1}{l_2}(-\omega^2 A_1) + \frac{g}{l_2} A_2 = 0$$

$$\left(-\omega^2 + \frac{g}{l_2}\right) A_2 - \omega^2 \frac{l_1}{l_2} A_1 = 0$$

(or) $$\frac{A_1}{A_2} = \frac{-\omega^2 + \frac{g}{l_2}}{\omega^2 \frac{l_1}{l_2}}.$$

$$-\omega^2 A_1 + \frac{m_2 l_2}{(m_1+m_2) l_1}(-\omega^2 A_2) + \frac{g}{l_1} A_1 = 0.$$

$$\left(-\omega^2 + \frac{g}{l_1}\right) A_1 - \frac{m_2 l_2 \omega^2}{(m_1+m_2) l_1} A_2 = 0.$$

$$\frac{A_1}{A_2} = \frac{m_2 l_2 \omega^2}{(m_1+m_2) l_1 \left(-\omega^2 + \frac{g}{l_1}\right)}$$

$$\frac{-\omega^2 + \frac{g}{l_2}}{\omega^2 \frac{l_1}{l_2}} = \frac{m_2 l_2 \omega^2}{(m_1+m_2) l_1 \left(-\omega^2 + \frac{g}{l_1}\right)}$$

$$m_2 \omega^4 = \left(-\omega^2 + \frac{g}{l_2}\right)(m_1+m_2)\left(-\omega^2 + \frac{g}{l_1}\right)$$

$$\omega^4 = \left(\frac{m_1+m_2}{m_2}\right)\left(\omega^4 - \omega^2\frac{g}{\lambda_1} - \omega^2\frac{g}{\lambda_2} + \frac{g^2}{\lambda_1\lambda_2}\right) = 0$$

$$\omega^4 - \frac{m_1+m_2}{m_2}\left(\omega^4 - \omega^2\frac{g}{\lambda_1} - \omega^2\frac{g}{\lambda_2} + \frac{g^2}{\lambda_1\lambda_2}\right) = 0.$$

$$\omega^4\left(1 - \frac{m_1+m_2}{m_2}\right) + \frac{m_1+m_2}{m_2}\omega^2 g\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) - \left(\frac{m_1+m_2}{m_2}\right)\frac{g^2}{\lambda_1\lambda_2} = 0$$

$$\omega^4 - \left(\frac{m_1+m_2}{m_1}\right)\omega^2\frac{(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}g + \frac{(m_1+m_2)g^2}{m_1\lambda_1\lambda_2} = 0.$$

$m_1 = m_2 = 5\,kg$

$\lambda_1 = \lambda_2 = 25\,cm$

$$\omega^4 - \frac{2m}{m}\omega^2\frac{2\lambda}{\lambda^2}g + \frac{2mg^2}{m\lambda^2} = 0$$

$$\omega^4 - 4\omega^2\frac{g}{\lambda} + \frac{2g^2}{\lambda^2} = 0.$$

$\lambda = 0.25\,m$

$$\omega^4 - \frac{4\times\omega^2\times 9.8}{0.25} + \frac{2\times 9.8^2}{(0.25)^2} = 0$$

$$\omega^4 - 156.8\,\omega^2 + 3073.28 = 0$$

$$\omega^2 = \frac{156.8 \pm \sqrt{(156.8)^2 - 4\times 3073.28}}{2}$$

$$\omega_{1,2}^2 = \frac{156.8 \pm 110.87}{2}$$

$\omega_1 = 11.56$ rad/sec

$\omega_2 = 4.8$ rad/sec.