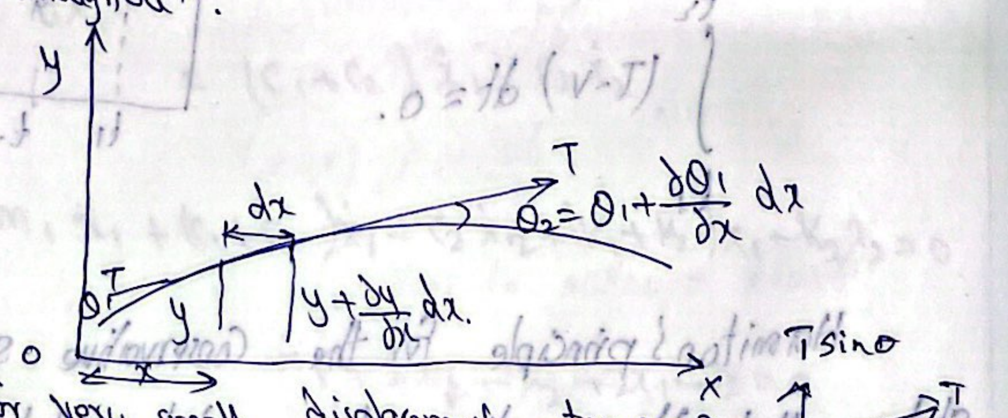
LATERAL VIBRATIONS OF A STRING:

Consider a vibrating string of mass ρ per unit length having transverse vibrations under Tension T as shown.

It is assumed that for a very small amplitude of string vibration the Tension T remains constant throughout.



for very small displacement $\tan\theta_1 = \theta$

$$\tan\theta_1 = \frac{\partial y}{\partial x}$$

$$\theta_1 = \frac{\partial y}{\partial x}$$

$$\theta_2 = \frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) dx$$

$$\theta_2 = \theta_1 + \frac{\partial \theta_1}{\partial x} dx.$$

Resolving Tension along y-axis.

$$T \sin\left[\theta_1 + \frac{\partial \theta_1}{\partial x} dx\right] - T\sin\theta_1 = \text{mass} \times \text{acceleration}$$

$$= \rho\, dx \frac{\partial^2 y}{\partial t^2}$$

$$T\left(\theta_1 + \frac{\partial \theta_1}{\partial x} dx\right) - T\theta_1 = \rho\, dx \frac{\partial^2 y}{\partial t^2}$$

$$T \frac{\partial \theta_1}{\partial x} dx = \rho\, dx \frac{\partial^2 y}{\partial t^2}$$

$$T \frac{\partial \theta_1}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2}$$

$$\frac{T}{\rho}\left(\frac{\partial \theta_1}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2} \qquad \theta_1 = \frac{\partial y}{\partial x}$$

$$\frac{T}{\rho} \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2}$$

$$\frac{T}{\rho}\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial^2 y}{\partial t^2}$$

Assuming $a^2 = \frac{T}{\rho}$, the above equation can be written as.

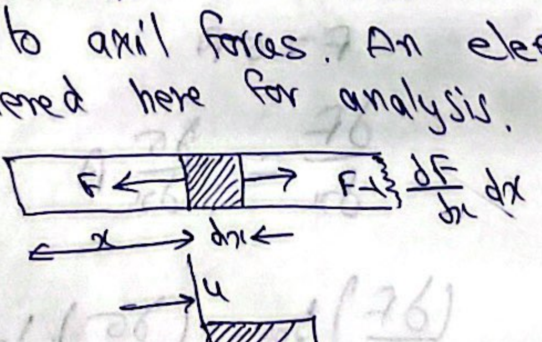
$$a^2\left(\frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 y}{\partial t^2}$$

This is one-dimensional wave equation for lateral vibrations of String.

LONGITUDINAL VIBRATIONS OF BARS

Let us consider thin and uniform bar for longitudinal vibrations as show. The bar is subjected to axil forces. An element dx of the bar is considered here for analysis.



if u is displacement at a distance x from left and it becomes $u + \frac{\partial u}{\partial x} dx$ @ $x + dx$.

$$dx + \frac{\partial u}{\partial x} dx - dx = \frac{\partial u}{\partial x} dx.$$

So strain of the element

$$\epsilon = \frac{\frac{\partial u}{\partial x} . dx}{dx}$$

$$= \frac{\partial u}{\partial x}.$$

A = cross sectional area of bar

ρ = density of the material

E = modulus of elasticity of the material

F = Force.

Net force.

$$\left(F + \frac{\partial F}{\partial x} dx\right) - F = m\, a$$

$$= dm \times \frac{\partial^2 u}{\partial t^2}.$$

$$\frac{\partial F}{\partial x} dx = (\rho\, dx\, A) \frac{\partial^2 u}{\partial t^2}$$

we know $\sigma = \frac{F}{A}$

$$F = \sigma A$$

$$\frac{\partial F}{\partial x} = \frac{\partial \sigma}{\partial x} A$$

$$\left(\frac{\partial F}{\partial x}\right) dx = \left(\frac{\partial \sigma}{\partial x}\right) dx\, A$$

$$\left(\frac{\partial \sigma}{\partial x}\right) dx\, \cancel{A} = \rho\, dx\, \cancel{A} \left(\frac{\partial^2 u}{\partial t^2}\right)$$

$$\frac{\partial \epsilon . E dx}{\partial x} = \rho\, dx \left(\frac{\partial^2 u}{\partial t^2}\right)$$

$$\frac{\sigma}{\epsilon} = E$$

$$\sigma = \epsilon E$$

$$\frac{\partial \sigma}{\partial x} = \frac{\partial \epsilon}{\partial x} E$$

$$\left(\frac{\partial \sigma}{\partial x}\right) dx\, A = \left(\frac{\partial \epsilon}{\partial x}\right) dx\, A E$$

$$\therefore \quad \frac{\partial \epsilon}{\partial x}\, dx\, AE = \rho\, dx\, A \left(\frac{\partial^2 u}{\partial t^2}\right)$$

$$\frac{E}{\rho} \frac{\partial \epsilon}{\partial x} = \frac{\partial^2 u}{\partial t^2}$$

$$\left(\frac{E}{\rho}\right) \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{E}{\rho}\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2 u}{\partial t^2} \qquad a^2 = \frac{E}{\rho}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$$

This is 1D wave equation.