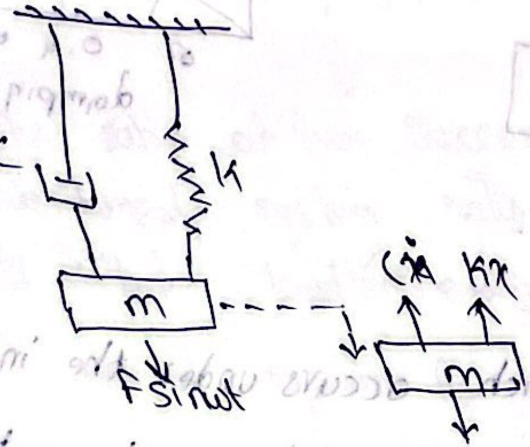
Equations of Motion with Harmonic Force. (16 marks)

Consider a spring mass with viscous damping while a harmonic force of frequency (ω) and amplitude F acts on it.



The equation of motion.

$$m\ddot{x} + c\dot{x} + kx = F\sin\omega t \longrightarrow (1)$$

This (1) is second order linear differential equation with constant coefficient. The general solution of the above equation.

$$x = x_c + x_p$$

x_c = Complementary solution

x_p = particular solution.

x_c is the solution of the homogeneous equation

$$m\ddot{x} + c\dot{x} + kx = 0$$

which we have already discussed in the previous chapter.

A Solution

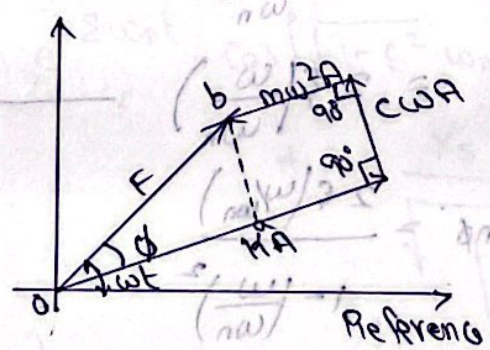
$$x_p = A\sin(\omega t - \phi) \longrightarrow (1.2)$$

assumed, where, A – amplitude

ϕ – phase with respect to harmonic force

$$\dot{x}_p = \omega A \cos(\omega t - \phi)$$
$$= \omega A \sin(\omega t - \phi + \pi/2)$$
$$\ddot{x}_p = \omega^2 A \sin(\omega t - \phi + \pi)$$
$$m\omega^2 A \sin(\omega t - \phi + \pi) + C\omega A \sin(\omega t - \phi + \pi/2) + KA \sin(\omega t - \phi) - F \sin\omega t = 0 \longrightarrow (2)$$

However, x_c vanishes because of damping with time.
From the eqn (2) represents four force namely inertia force, damping force, spring force and harmonic force. By the application of these force, the system is supposed to be in equilibrium. The vector diagram coming out from this equation (2) shows that spring force is perpendicular to damping force and damping force is perpendicular to inertia force.



from Δ oab

$$F^2 = (KA - m\omega^2 A)^2 + (C\omega A)^2 \longrightarrow (3)$$

$$A = \frac{F}{\sqrt{(K - m\omega^2)^2 + (C\omega)^2}}$$

$$= \frac{F/K}{\sqrt{\left[\frac{K}{K} - \frac{m\omega^2}{K}\right]^2 + \left(\frac{C\omega}{K}\right)^2}}$$

$\omega_n^2 = K/m$

$$A = \frac{F/K}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left(\frac{C\omega}{K}\right)^2}} \longrightarrow (4)$$

$\Delta\, oab$

$$\tan\phi = \frac{ab}{oa}$$

$$= \frac{C\omega A}{KA - m\omega^2 A}$$

$$= \frac{C\omega}{K - m\omega^2}$$

$$\tan\phi = \frac{C\omega/K}{1 - \frac{\omega^2}{\omega_n^2}} \longrightarrow (5)$$

We know that

$$\frac{C\omega}{K} = \left(\frac{c}{c_c}\right)\left(\frac{c_c}{2m}\right)\left(\frac{2m}{K}\right)\cdot\omega$$

$$= \xi\, \omega_n \frac{2}{\omega_n^2}\, \omega$$

$$= 2\xi\left(\frac{\omega}{\omega_n}\right) \longrightarrow (6)$$

sub (6) in (5)

$$\tan\phi = \frac{2\xi(\omega/\omega_n)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

equ (4) can be written

$$A = \frac{F/K}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

Let us assume that $F/K = X_s$ where X_s is called Zero frequency deflection.

$$A = \frac{X_s}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$\frac{A}{X_s} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \longrightarrow (7)$$

This equ (7) is non-dimensional quantity A/X_s is known as magnification factor or amplitude ratio.

So that particular solution of equ (1.1) can be written as.

$$x_p = X_s \frac{\sin(\omega t - \phi)}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + (2\xi\omega/\omega_n)^2}} \longrightarrow (8)$$

The complete solution can be written as.

$$x = x_c + x_p$$

$$= A_1 e^{-\xi\omega_n t}\cos\left(\sqrt{1-\xi^2}\,\omega_n t + \phi_1\right) + \frac{X_s \sin(\omega t - \phi)}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \longrightarrow (9)$$

The value A_1 and ϕ_1 can be determined form initial condition.

The frequency at which maximum amplitude occurs can be obtained by differentiating (7) with respect to (ω/ω_n) and equating the differential to zero.

$$\frac{d(A/X_s)}{d(\omega/\omega_n)} = \frac{2\left[1-(\omega/\omega_n)^2\right]\left[-2(\omega/\omega_n)\right] + 2\left[2\xi\omega/\omega_n\right](2\xi)}{2\left\{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\xi\omega/\omega_n\right]^2\right\}^{3/2}} = 0$$

which leads to $\frac{\omega_{max}}{\omega_n} = \sqrt{1-2\xi^2} \neq \omega_{peak} = \omega_n\sqrt{1-2\xi^2}$

where ω_{max} is the frequency corresponding to the max Am