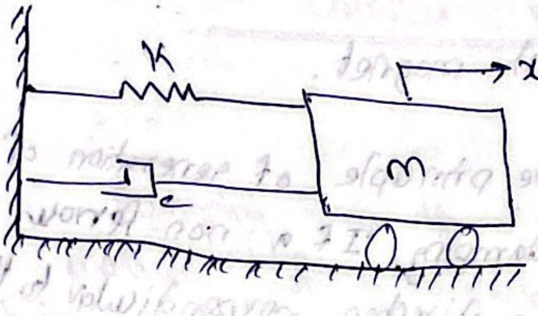
Differential equations of Damped free vibration:



$$c\frac{dx}{dt} = c\dot{x} \rightarrow \text{Damping force.}$$

$$m\frac{d^2x}{dt^2} = m\ddot{x} = \text{acceleration force}$$

$$kx = \text{spring force}$$

Thus the equation of motion,

$$m\ddot{x} + c\dot{x} + kx = 0.$$

Free body diagram

Assuming. $x = e^{ut}$

$$\dot{x} = ue^{ut}$$

$$\ddot{x} = u^2e^{ut}.$$

$$\therefore mu^2e^{ut} + cue^{ut} + ke^{ut} = 0. \text{ [quadratic Equation]}$$

$$\boxed{mu^2 + cu + K = 0}$$

$$ax^2 + bx + c = 0, \quad = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$\therefore u = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

$$u_1 = \frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}}$$

$$u_2 = \frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}.$$

Now the solution of equ.

$$x = A_1e^{u_1t} + A_2e^{u_2t}$$

$$\therefore x = A_1e^{\left[\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}\right]t} + A_2e^{\left[-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}\right]t}$$

Critical Damping Constant and Damping ratio

The critical damping c_c is defined as the value of damping coefficient c. for which mathematical term $\left(\frac{c}{2m}\right)^2 - \frac{k}{m}$ in eqn is equal to zero.

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0. \qquad \frac{c_c}{2m} = \sqrt{\frac{k}{m}}$$

$$c_c = 2m\sqrt{k/m}$$
$$= 2m\omega.$$

Ratio of c to c_c is termed as damping ratio.

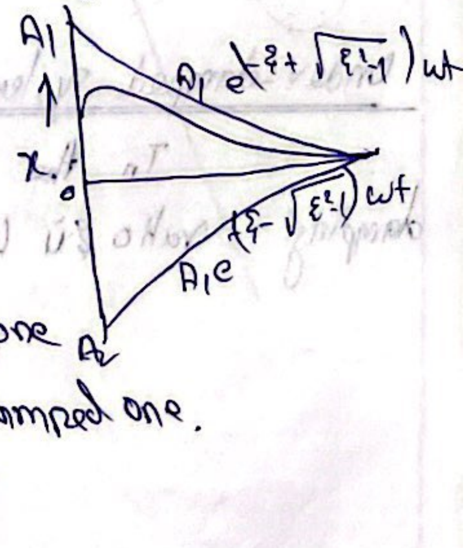
$$\xi = \frac{c}{c_c}$$

$$\frac{c}{2m} \text{ of } \cancel{\text{eqn}}. = \left(\frac{c}{c_c}\right)\left(\frac{c_c}{2m}\right) = \xi\omega.$$

$$x = A_1 e^{\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega t} + A_2 e^{\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega t}$$

nature of the system depends upon the value of damping. Depending upon the value damping ratio ξ, the damped system.

$\xi > 1$
$\xi = 1$
$\xi < 1.$



Over damped system.

$$x = A_1 + A_2.$$

Damping ratio ξ is more than one $\xi > 1$ the system is known as over-damped one.

Critically damped system:-

$$\xi = 1$$

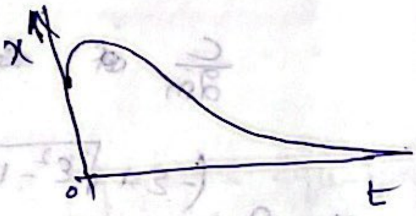
$$\frac{c}{2m} = \sqrt{\frac{k}{m}}$$

$$u_1 = u_2 = -\xi\omega$$
$$= -\omega$$

$$\therefore x = A_1 e^{-\omega t} + A_2 t e^{-\omega t}$$

$$= (A_1 + A_2 t) e^{-\omega t}$$

initial condition,



$$x_0 = A_1$$
$$\dot{x}_0 = A_1(-\omega) + A_2$$
$$A_2 = \dot{x}_0 + \omega x_0$$

The value of x decrease as t increases and finally become zero as t tends to infinity. This is also an aperiodic motion

Under-damped system:-

In the case of under-damped system the damping ratio ξ is less than unity.

$$u_1 = \left[-\xi + j\sqrt{1-\xi^2}\right]\omega$$

$$u_2 = \left[-\xi - j\sqrt{1-\xi^2}\right]\omega$$

$$j = \sqrt{-1}$$

$$x = A_1 e^{\left[-\xi + j\sqrt{1-\xi^2}\right]\omega t} + A_2 e^{\left[-\xi - j\sqrt{1-\xi^2}\right]\omega t}$$

$$= e^{-\xi\omega t}\left[A_1 e^{j\sqrt{1-\xi^2}\,\omega t} + A_2 e^{-j\sqrt{1-\xi^2}\,\omega t}\right].$$

$e^{jx} = \cos x + j \sin x$.

$$= e^{-\xi \omega t}\left[A_1 \cos\sqrt{1-\xi^2}\,\omega t + A_1 j \sin\sqrt{1-\xi^2}\,\omega t + A_2 \cos\sqrt{1-\xi^2}\,\omega t - A_2 j \sin\sqrt{1-\xi^2}\,\omega t\right].$$

$$= e^{-\xi \omega t}\left[C_1 \cos\sqrt{1-\xi^2}\,\omega t + C_2 \sin\sqrt{1-\xi^2}\,\omega t\right].$$

$C_1 = A_1 + A_2$

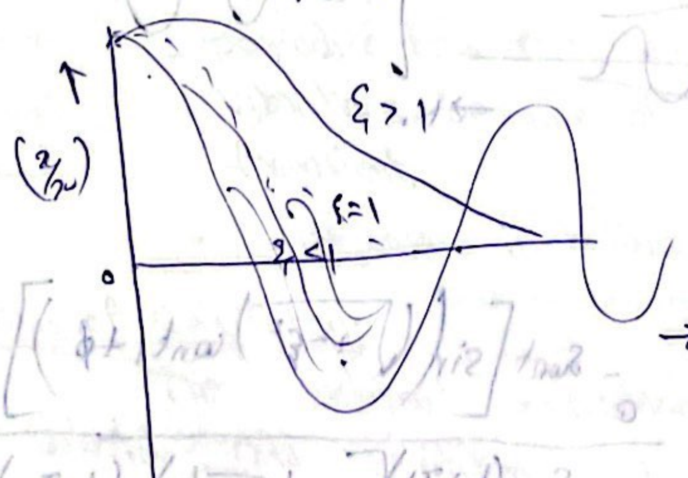
$C_2 = (A_1 - A_2)\,j$

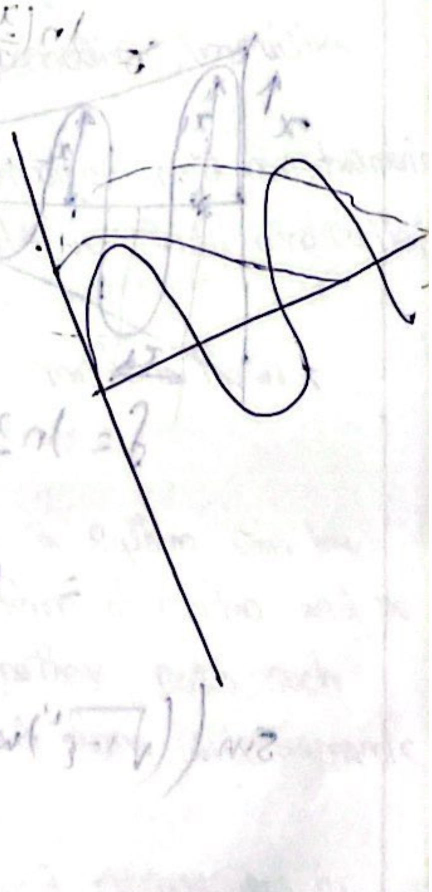
$$x = C_3 e^{-\xi \omega t} \sin\left(\sqrt{1-\xi^2}\,\omega t + \phi_1\right)$$

$$x = C_4 e^{-\xi \omega t} \cos\left(\sqrt{1-\xi^2}\,\omega t + \phi_2\right)$$

$C_1, C_2, C_3, C_4, \phi_1$ and ϕ_2 are arbitrary.

$\omega_d = \omega\sqrt{1-\xi^2}$.





$\left[\phi + \left(\omega t + \frac{2\pi}{\omega\sqrt{1-\xi^2}}\right)\right]$ (or) $\frac{2\pi}{\omega_d} = t_d$.