TAYLOR SERIES METHOD:

Consider the first order differential eqn

$$\frac{dy}{dx} = f(x,y) \quad \text{with } y(x_0) = y_0.$$

Hence the Taylor's series expansion of $y(x)$ is given by

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots\cdots$$

Let $x_1 = x_0 + h$

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \cdots\cdots$$

Now let $x_2 = x_1 + h$

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \cdots$$

(1) Using Taylor Series method find y at $x=0.1$ if $\frac{dy}{dx} = x^2y - 1$, $y(0)=1$

Soln: Gn: $y' = x^2y - 1$

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

Taylor series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \dots$$

$$y' = x^2y - 1 \Rightarrow y_0' = -1$$

$$y'' = 2xy + x^2y' \Rightarrow y_0'' = 0$$

$$y''' = 2xy' + 2y + 2xy' + x^2y'' \Rightarrow y_0''' = 2$$

$$= 2y + 4xy' + x^2y''$$

$$y^{iv} = 2y' + 4xy'' + 4y' + x^2y''' + 2xy'' \Rightarrow y_0^{iv} = -6$$

$$= 6y' + 6xy'' + x^2y'''$$

Now $y_1 = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(-6) + \dots$

$= 1 - 0.1 + 0.00033 - 0.000025$

$= 0.900305$

Alternate Method :

$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \frac{(x-x_0)^4}{4!} y_0'''' + \dots$

$= 1 + (x-0)(-1) + \frac{(x)^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \dots$

$= 1 - x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{x^4}{4!}(-6) + \dots$

$y(0.1) = 1 - 0.1 + \frac{(0.1)^2}{2!} + 2\frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!}(-6) + \dots$

$= 0.900305$

(2) Solve $y' = x + y$; $y(0) = 1$ by Taylor's series method. Find the values y at $x = 0.1$ and $x = 0.2$.

Soln:

$$y' = x + y$$

$$x_0 = 0 \; ; \; y_0 = 1 \quad h = 0.1$$

Taylor series is

$$y(x) = y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

$y' = x + y \Rightarrow y_0' = 1$

$y'' = 1 + y' \Rightarrow y'' = 2$

$y''' = y'' \Rightarrow y''' = 2$

$y^{iv} = y''' \Rightarrow y^{iv} = 2$

$$y = 1 + x(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(2) + \cdots$$

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \cdots$$

$$y(0.1) = 1+(0.1)+(0.1)^2+\frac{(0.1)^3}{3}+\frac{(0.1)^4}{12}+\cdots$$

$$= 1+0.1+0.01+0.00033+0.00000833$$

$$= 1.1103$$

$$y(0.2) = 1+(0.2)+(0.2)^2+\frac{(0.2)^3}{3}+\frac{(0.2)^4}{12}+\cdots$$

$$= 1+0.2+0.04+0.00267+0.00013$$

$$= 1.2428.$$

(3) Using Taylor series method find y at $x=0.1$ correct to 4 decimal places from $\frac{dy}{dx} = x^2-y$, $y(0)=1$ with $h=0.1$.

Soln: 0.905163

(4) Using Taylor method, compute $y(0.2)$ & $y(0.4)$ correct to 4 decimal places yn. $y' = 1-2xy$ and $y(0)=0$.

Soln: $0.2 \rightarrow 0.194752003$

$0.4 \rightarrow 0.359883723$