DERIVATIVES FROM INTERPOLATION

NEWTON'S FORWARD

$$y(x) = P_n(x) = f(x)$$

$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots + \frac{u(u-1)(u-2)\dots(u-(n-1))}{n!}\Delta^n y_0$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \dots\right]$$

putting $x = x_0$, then $u = 0$ and above eqn. reduces to

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6u-6}{3!}\Delta^3 y_0 + \frac{12u^2-36u+22}{4!}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\frac{6}{3!}\Delta^3 y_0 + \frac{24u-36}{4!}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \frac{7}{4}\Delta^5 y_0\right]$$

NEWTON'S BACKWARD

$$y(x) = P_n(x) = f(x)$$

$$= y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots + \frac{u(u+1)(u+2)\cdots(u+(n-1))}{n!}\nabla^n y_n$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{4!}\nabla^4 y_n + \cdots\right]$$

At $x = x_n$, $u = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6u+6}{3!}\nabla^3 y_n + \frac{12u^2+36u+22}{4!}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \frac{24u+36}{4!}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \frac{7}{4}\nabla^5 y_n\right]$$

① Find $f'(3)$ and $f''(3)$ for the following data:

x :	3.0	3.2	3.4	3.6	3.8	4.0
$f(x)$:	−14	−10.032	−5.296	−0.256	6.672	14

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3.0	−14					
		3.968				
3.2	−10.032		0.768			
		4.736		−0.464		
3.4	−5.296		0.304		2.048	
		5.04		1.584		−5.12
3.6	−0.256		1.888		−3.072	
		6.928		−1.488		
3.8	6.672		0.4			
		7.328				
4.0	14					

By Newton's forward formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0}$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

Here $h = 0.2$

$$= \frac{1}{0.2}\left[3.968 - \frac{1}{2}(0.768) + \frac{1}{3}(-0.464) - \frac{1}{4}(2.048) + \frac{1}{5}(-5.12)\right]$$

$$= \frac{1}{0.2}\left[3.968 - 0.384 - 0.1547 - 0.512 - 1.024\right]$$

$$= \frac{1}{0.2}\left[1.8933\right]$$

$$= 9.4665$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$= \frac{1}{(0.2)^2}\left[0.768 - (-0.464) + \frac{11}{12}(2.048) - \frac{5}{6}(-5.12)\right]$$

$$= \frac{1}{0.04}\left[0.768 + 0.464 + 1.8773 + 4.267\right]$$

$$= \frac{1}{0.04}\left[7.3763\right] = ~~184.4075~~ \; 36.875$$

The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the entire data

Time (sec) :	0	5	10	15	20
Velocity (m/sec) :	0	3	14	69	228.

Soln: Velocity $\rightarrow v$

Acceleration $\rightarrow \frac{dv}{dt}$

Initial acceleration $= \frac{dv}{dt}$ at $t=0$

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
		3			
5	3		8		
		11		36	
10	14		44		24
		55		60	
15	69		104		
		159			
20	228				

Initial acceleration $\frac{dv}{dt}$ at $t=0$ is

$$= \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \cdots\right]$$

$h=5$

$$= \frac{1}{5}\left[3 - \frac{1}{2}\times 8 + \frac{1}{3}\times 36 - \frac{1}{4}\times 24\right]$$

$$= \frac{1}{5}\left[3-4+12-6\right]$$

$$= \frac{1}{5}(5)$$

$$= 1$$

(5) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 51$ from the following data:

x :	50	60	70	80	90
y :	19.96	36.65	58.81	77.21	94.61

Soln: Here $h = 10$, $x_0 = 50$, $x = 51$

$$\therefore u = \frac{x - x_0}{h} = \frac{51-50}{10} = 0.1$$

$$\left(\frac{dy}{dx}\right)_{x=51} = \left(\frac{dy}{dx}\right)_{u=0.1} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0 + \cdots\right]$$

x	$u=\frac{x-x_0}{10}$, $x_0=50$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	0	19.96				
			16.69			
60	1	36.65		5.47		
			22.16		-9.23	
70	2	58.81		-3.76		11.99
			18.4		2.76	
80	3	77.21		-1.00		
			17.4			
90	4	94.61				

$$\left(\frac{dy}{dx}\right)_{u=0.1} = \frac{1}{10}\left[16.69 + \frac{2\times 0.1-1}{2}\times 5.47 + \frac{3\times 0.1^2-6\times 0.1+2}{6}(-9.23) + \frac{4\times 0.1^3-18\times 0.1^2+22\times 0.1-6}{24}\times 11.99\right]$$

$= \frac{1}{10}[16.69 - 2.188 - 2.1998 - 1.9863]$

I 1.03159

(2) Find the first, second & third derivatives of $f(x)$ at $x = 1.5$ if

x :	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$:	3.375	7.00	13.625	24.0	38.875	59.0

Soln: $f'(1.5) = 4.75$; $f''(1.5) = 9$; $f'''(1.5) = 6$