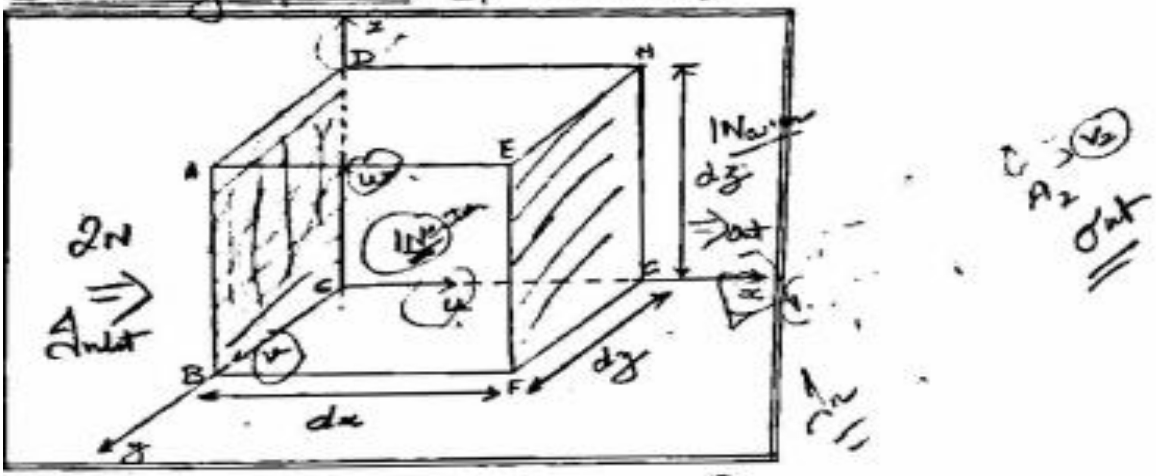
Continuity Equation : [3 - Dimensions]



What is Continuity equation?

⇒ The equation based on the Principle of Conservation of mass.

3-Dimension Derivation:

⇒ dx, dy, dz is length in direction x, y, z.

⇒ u is inlet velocity component in x direction
v is inlet velocity component in y direction
w is inlet velocity component in z direction

⇒ Mass of fluid entering the face ABCD per sec

$$\frac{m}{s} = \frac{\rho \times V}{s}$$ $m = \rho \times$ Volume, $\boxed{m = \rho \times V}$

$$= \frac{\rho \times A \times l}{s} \Rightarrow \rho A V$$ $V = A \times l$, $\frac{l}{s} = V$

$= \rho \times$ velocity in x direction $\times$ Area of ABCD

$= \rho \times u \times (dy \times dz)$

⇒ Mass of fluid leaving the face EFGH per second

$$= \rho \times u \times dy \times dz + \underbrace{\frac{\partial}{\partial x}(\rho u\, dy\, dz)\, dx}_{\text{change of mass at } x\text{-direction}}$$

⇒ Rate of increase in mass

$$= \text{mass through ABCD} - \text{Mass through EFGH}$$

$$= \rho u\, \cancel{dy}\, dz - \rho u\, \cancel{dy}\, dz - \frac{\partial}{\partial x}(\rho u\, dy\, dz)\, dx$$

$$= -\frac{\partial}{\partial x}(\rho u)\, dy\, dz\, dx$$

$$= -\frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz \checkmark$$

Similarly,

Rate of increase in mass y - direction

$$= -\frac{\partial}{\partial y}(\rho v)\, dx\, dy\, dz \checkmark$$

Rate of increase in mass z - direction

$$= -\frac{\partial}{\partial z}(\rho w)\, dx\, dy\, dz \checkmark$$

∴ Total rate of increase in mass

$$= -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dx\, dy\, dz. \quad \rightarrow (1)$$

$$\boxed{\therefore \text{Net Gain} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dx\, dy\, dz}$$

Note:- (1) By law of Conservation of mass, there is no accumulation of mass i.e., mass is neither Created nor destroyed in the fluid element.

(2) So the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element.

Mass of fluid in element $= \rho(dx\,dy\,dz)$

$$= \frac{\partial}{\partial t}(\rho\,dx\,dy\,dz)$$

$$= \frac{\partial \rho}{\partial t}(dx\,dy\,dz) \rightarrow (2)$$

$\rho = \frac{m}{V}$, $m = \rho \times V$

Equating (1) & (2)

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] \cancel{dx\,dy\,dz} = \frac{\partial \rho}{\partial t}(\cancel{dx\,dy\,dz})$$

$$\therefore \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$\Rightarrow$ For steady flow $\boxed{\frac{\partial \rho}{\partial t} = 0}$ & hence the eqn becomes,

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$\Rightarrow$ If the fluid is incompressible, then ρ is constant & equation becomes.

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \rightarrow \text{Continuity of eqn 3-Dimension}$$