Type - III

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots$

1. Solve $(D^2+2)y = x^2$.

AE $\Rightarrow$ $m^2+2=0$ $\quad m^2 = -2$

$m = \pm\sqrt{2}i$

CF $= e^{0x}(A\cos 2x + B\sin 2x)$

$= A\cos\sqrt{2}x + B\sin\sqrt{2}x$.

PI $\Rightarrow \frac{1}{D^2+2}x^2$ [constant term should be 1]

$= \frac{1}{2(D^2/2+1)}x^2 = \frac{1}{2}(1+D^2/2)^{-1}x^2$

$= \frac{1}{2}\left[1 - \frac{D^2}{2} + \left(\frac{D^2}{2}\right)^2 - \dots\right]x^2$ $\quad \hookrightarrow D^3=0$.

$= \frac{1}{2}\left[1 - \frac{D^2}{2}\right]x^2 = \frac{1}{2}\left[x^2 - \frac{D^2}{2}x^2\right]$

$= \left[\frac{x^2}{2} - \frac{1}{4}(2)\right] = \frac{x^2}{2} - \frac{1}{2} = \frac{1}{2}(x^2-1)$

$y = CF + PI$.

$= A\cos\sqrt{2}x + B\sin\sqrt{2}x + \frac{1}{2}(x^2-1)$

2. $(D^2+3D+2)y = x^2$.

AE $\Rightarrow$ $m^2+3m+2=0$ $\quad (m+2)(m+1)=0$

$m = -2, -1$

CF $= Ae^{-x} + Be^{-2x}$.

$$PI = \frac{1}{D^2+3D+2}x^2 = \frac{1}{2\left(1+\frac{D^2+3D}{2}\right)}x^2$$

$$= \frac{1}{2}\left(1+\frac{D^2+3D}{2}\right)^{-1}x^2$$

$$= \frac{1}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2-\cdots\right]x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{9D^2}{4}\right]x^2$$

$$= \frac{1}{2}\left[x^2-\frac{2}{2}-\frac{3(2x)}{2}+\frac{9.2}{4}\right]$$

$$= \frac{1}{2}\left[x^2-1-\frac{3x}{\cancel{2}}+\frac{9}{2}\right] = \frac{1}{2}\left[x^2-\frac{3x}{\cancel{2}}+\frac{7}{2}\right]$$

$y = CF + PI$

$$= Ae^{-x}+Be^{-2x}+\frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]$$

Type - iv $\Rightarrow e^{ax}\phi(x)$

1. $(D^2-4D+3)y = e^x\cos 2x$

AE $\Rightarrow m^2-4m+3=0. \Rightarrow (m-3)(m-1)=0$

$m=1,3, \Rightarrow$ Roots are real and different

$CF = Ae^{x}+Be^{3x}$

$$PI = \frac{1}{D^2-4D+3}e^x\cos 2x$$

$D \Rightarrow D+1$

$$= e^x\frac{1}{(D+1)^2-4(D+1)+3}\cos 2x = e^x\frac{1}{D^2+2D+1-4D-4+3}\cos 2x$$

$D^2 \to -(a^2) \to -4$

$$= e^x\frac{1}{D^2-2D}\cos 2x = e^x\frac{1}{-4-2D}\cos 2x$$

$$= e^x\frac{1}{-2D-4}\cos 2x = e^x\frac{1}{-2D-4}\left(\frac{-2D+4}{-2D+4}\right)\cos 2x$$

$$= e^{x} \frac{-2D+4}{4D^2-16} \cos 2x = e^{x} \frac{\left[-2D\cos 2x + 4\cos 2x\right]}{-16-16}$$

$$= \frac{-e^{x}}{32}\left[+2\sin 2x(2) + 4\cos 2x\right]$$

$$= \frac{-e^{x}}{32}\left[4\sin 2x + 4\cos 2x\right] = \frac{-4e^{x}}{32}\left[\sin 2x + \cos 2x\right]$$

$$= \frac{-e^{x}}{8}\left[\sin 2x + \cos 2x\right]$$

$y = CF + PI$

$$= Ae^{x} + Be^{2x} - \frac{e^{x}}{8}\left[\sin 2x + \cos 2x\right]$$

Type – 5 RHS : $x\sin x$

PI : $x\frac{1}{\phi(D)}\sin x - \frac{\phi'(D)}{(\phi(D))^2}\sin x$

1. Solve $(D^2+4)y = x\sin x$

Auxiliary Equation $\Rightarrow m^2+4=0$ $m^2=-4$ $m=\pm 2i$

The roots are imaginary $\Rightarrow CF = e^{0x}(A\cos 2x + B\sin 2x)$

$$= A\cos 2x + B\sin 2x$$

Particular Integral :

$$x\frac{1}{D^2+4}\sin x - \frac{2D}{(D^2+4)^2}\sin x$$

$D^2 \to -a^2 = -1$

$$x\frac{1}{-1+4}\sin x - \frac{2D\sin x}{(-1+4)^2}$$

$PI \Rightarrow \frac{x}{3}\sin x - \frac{2\cos x}{9} \Rightarrow y = CF + PI$

$$= A\cos 2x + B\sin 2x + \frac{x}{3}\sin x - \frac{2}{9}\cos x$$

2. $(D^2-2D+1)y = x\sin x$

AE ⇒ $m^2-2m+1=0$ $\quad m = 1,1$

Roots are Real and equal

$CF = (A+Bx)e^x$.

$$PI = x\frac{1}{D^2-2D+1}\sin x - \frac{(2D-2)}{(D^2-2D+1)^2}\sin x$$

$D^2 \to -a^2 = -1$

$$= x\frac{1}{\cancel{-1}-2D+\cancel{1}}\sin x - \frac{(2D-2)\sin x}{(\cancel{-1}-2D+\cancel{1})^2}$$

$$= -x\frac{1}{2D}\left(\frac{2D}{2D}\right)\sin x - \frac{(2D-2)}{(2D)^2}\sin x$$

$$= \frac{-x\,2D\sin x}{4D^2} - \frac{(2D-2)\sin x}{4D^2}$$

$$= \frac{-x.2D\sin x}{-4} - \frac{(2D-2)\sin x}{-4}$$

$$= \frac{x}{2}\cos x + \frac{2D\sin x}{4} - \frac{2\sin x}{4}$$

$$= \frac{x}{2}\cos x + \frac{1}{2}\cos x - \frac{1}{2}\sin x$$