Legendre's Linear Differential equation

$\hookrightarrow (ax+b)y''$

Transform the equation to constant Coefficient ~~of~~ equation

(ax+b)

① $(2x+3)^2 y'' - (2x+3)y' + 2y = 6x.$

put $(2x+3) = e^z$

$2x = e^z - 3$

$x = \frac{e^z - 3}{2}$

$z = \log(2x+3)$

$(2x+3)^2 D^2 = 2^2(\theta^2 - \theta) = 4(\theta^2 - \theta)$

$x^2 D^2 = D'^2 - D'$

$(2x+3)D = 2\theta$

$xD = D'$

$[(2x+3)^2 D^2 - (2x+3)D + 2]y = 6x$

$[4(\theta^2 - \theta) - 2\theta + 2]y = \overset{3}{\cancel{6}}\left(\frac{e^z - 3}{\cancel{2}}\right)$

$[4\theta^2 - 4\theta - 2\theta + 2]y = 3e^z - 9$

$[4\theta^2 - 6\theta + 2]y = 3e^z - 9$

Which is a linear equation with constant Coefficient

2. $(1+x)^2 y'' + (1+x)y' + y = 2\sin[\log(1+x)]$

$1+x = e^z$

$z = \log(1+x)$

$(1+x)D' = \theta$

$xD = 1\theta$

$(1+x)^2 D^2 = 1^2(\theta^2 - \theta) = \theta^2 - \theta$

$[(1+x)^2 D^2 + (1+x)D + 1]y = 2\sin[\log(1+x)]$

$[\theta^2 - \theta + \theta + 1]y = 2\sin z$

$[\theta^2 + 1]y = 2\sin z.$

Solve

3. $(2x+3)^2 y'' - (2x+3)y' - 12y = 6x.$

$2x+3 = e^z \Rightarrow z = \log(2x+3)$

$2x = e^z - 3$

$x = \frac{e^z - 3}{2}$

$(2x+3)D = 2\theta$

$(2x+3)^2 D^2 = 2^2(\theta^2 - \theta) = 4(\theta^2 - \theta)$

$[(2x+3)^2 D^2 - (2x+3)D - 12]y = 6x$

$[4(\theta^2 - \theta) - 2\theta - 12]y = 6\left[\frac{e^z - 3}{2}\right]$

$[4\theta^2 - 4\theta - 2\theta - 12]y = 6\left[\frac{e^z - 3}{2}\right]$

$[4\theta^2 - 6\theta - 12]y = 3(e^z - 3)$

$= 3e^z - 9$

$AE \Rightarrow 4m^2 - 6m - 12 = 0.$

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, $a = 4$, $b = -6$, $c = -12$

$$= \frac{6 \pm \sqrt{36-4(4)(-12)}}{2(4)} = \frac{6 \pm \sqrt{36+192}}{8}$$

$$= \frac{6}{8} \pm \frac{\sqrt{228}}{8} = \frac{3}{4} \pm \frac{2\sqrt{57}}{8} = \frac{3}{4} \pm \frac{\sqrt{57}}{4}$$

The Roots are Real and different

$$CF = Ae^{\left(\frac{3+\sqrt{57}}{4}\right)z} + Be^{\left(\frac{3-\sqrt{57}}{4}\right)z}$$

$$PI_1 = \frac{1}{4D^2-6D-12} 3e^z$$

$$= \frac{1}{4(1)^2-6(1)-12} 3e^z = \frac{+3e^z}{4-6-12} = \frac{-3e^z}{14}$$

$$PI_2 = \frac{1}{4D^2-6D-12} -9e^{0z} = \frac{-9}{-12} = \frac{3}{4}$$

$\therefore$ The solution is $y = CF + PI$

$$= Ae^{\left(\frac{3+\sqrt{57}}{4}\right)z} + Be^{\left(\frac{3-\sqrt{57}}{4}\right)z} - \frac{3}{14}e^z + \frac{3}{4}$$

Sub $z = \log(2x+3)$

$$y = A(2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right)} + B(2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right)} - \frac{3}{14}(2x+3) + \frac{3}{4}$$

4) $[(x+2)^2D^2 - (x+2)D + 1]y = 3x+4$

$x+2 = e^z \Rightarrow x = e^z - 2$

$z = \log(x+2)$

$(x+2)D = \theta \qquad (x+2)^2D^2 = 1^2(\theta^2 - \theta)$

$[\theta^2 - \theta - \theta + 1]y = 3(e^z - 2) + 4$

$(\theta^2 - 2\theta + 1)y = 3e^z - 6 + 4$

$(\theta^2 - 2\theta + 1)y = 3e^z - 2.$

AE $\Rightarrow m^2 - 2m + 1 = 0 \Rightarrow m = +1, 1$

Roots are real and equal

$CF = (A + Bz)e^z$

$PI_1 = \frac{1}{D^2 - 2D + 1} 3e^z = \frac{1}{(1)^2 - 2(1) + 1} 3e^z$

$= \frac{z}{2D - 2} 3e^z = \frac{z}{2(1) - 2} 3e^z$

$= \frac{z^2}{2} \cdot 3e^z = \frac{3}{2} z^2 e^z$

$PI_2 = \frac{1}{D^2 - 2D + 1}(-2)e^{0z} = -2$

$y = CF + PI_1 + PI_2$

$= (A + Bz)e^z + \frac{3}{2} z^2 e^z - 2$

$= [A + B(\log(x+2)](x+2) + \frac{3}{2}[\log(x+2)]^2(x+2) - 2.$