Euler's Cauchy Homogeneous Linear Equation

$\hookrightarrow x^2y''$

Procedure:-

Put $x = e^z$

$\log x = z$

$xD = D' \quad \theta'$

$x^2D^2 = D'^2 - D' \quad \theta'^2 - \theta'$

1. Solve by Cauchy Method → start with x^2

i) $x^2y'' + 2xy' = 0.$

$$(x^2D^2 + 2xD)y = 0 \quad \rightarrow (1)$$

put $x^2D^2 = D'^2 - D'$

$xD = D'$

$$(1) \Rightarrow (D'^2 - D' + 2D')y = 0.$$

$$(D'^2 + D')y = 0$$

A.E $\Rightarrow m^2 + m = 0 \Rightarrow m(m+1) = 0$

$m = 0, \ m = -1$

Roots are Real & different

$$CF = Ae^{0z} + Be^{-z}$$

$$= A + Be^{-z} = A + \frac{B}{e^z} = A + \frac{B}{x}$$

$PI = 0$

$y = CF + PI$

The soln is $y = A + \frac{B}{x}$

2. $(x^2D^2 - 3xD + 4)y = x^2$

Put $x^2D^2 = D'^2 - D'$ & $xD = D'$ $\qquad x = e^z$, $x^2 = e^{2z}$

$(D'^2 - D' - 3D' + 4)y = x^2$

$(D'^2 - 4D' + 4)y = e^{2z}$

AE $\Rightarrow m^2 - 4m + 4 = 0$

$m = 2, 2$

The Roots are Real & equal

$CF = (A + Bz)e^{2z}$

$$PI = \frac{1}{D'^2 - 4D' + 4} e^{2z}$$

$$= \frac{1}{(2)^2 - 4(2) + 4} e^{2z} = \frac{1}{0} e^{2z}$$

$$= \frac{z}{2D' - 4} e^{2z} = \frac{z}{4-4} e^{2z}$$

$$= \frac{z^2}{2} e^{2z} = \frac{(\log x)^2}{2} x^2$$

$$PI = \frac{(\log x)^2}{2} x^2$$

$y = CF + PI$

$$= (A + Bz)e^{2z} + \frac{z^2}{2} e^{2z}$$

$$= (A + B\log x)x^2 + \frac{(\log x)^2}{2} x^2$$

HW $\quad x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$

Ans $\quad y = \frac{A}{x} + \frac{1}{x^2}(A + e^x)$