Method of variation of parameters

Procedure:

To find the general solution of second order equation

$(D^2 - a_1 D + a_2)y = X$ [where X is a function of x]

Solve by method of variation of parameters

i) $(D^2+4)y = \sec 2x$

AE is $m^2+4=0 \Rightarrow m^2=-4 \Rightarrow m = \pm 2i$

The roots are complex numbers

$\therefore CF = e^{0x}(A\cos 2x + B\sin 2x)$

$= A\cos 2x + B\sin 2x$.

$f_1 = \cos 2x$; $f_2 = \sin 2x$.

$f_1' = -2\sin 2x$ $\quad f_2' = 2\cos 2x$

W = Wronskian of f_1 & f_2

$W = f_1 f_2' - f_1' f_2 = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$= 2\cos^2 2x + 2\sin^2 2x$

$= 2[\cos^2 2x + \sin^2 2x] = 2$

$\boxed{W = 2}$

$PI = Pf_1 + Qf_2$

$P = -\int \frac{f_2 X}{W}\,dx \qquad Q = \int \frac{f_1 X}{W}\,dx$

$$P = -\int \frac{f_2 x}{W}\,dx = -\int \frac{\sin 2x \sec 2x}{2}\,dx$$

$$= \frac{-1}{2}\int \sin 2x \cdot \frac{1}{\cos 2x}\,dx = \frac{-1}{2}\int \tan 2x\,dx$$

$$= \frac{-1}{2}\frac{\log(\sec 2x)}{2} = \frac{-1}{4}\log(\sec 2x)$$

$$Q = \int \frac{f_1 x}{W}\,dx = \int \frac{\cos 2x \sec 2x}{2}\,dx$$

$$= \frac{1}{2}\int \cancel{\cos 2x} \cdot \frac{1}{\cancel{\cos 2x}}\,dx = \frac{1}{2}\int dx$$

$$Q = \frac{x}{2}$$

$$PI = \frac{-1}{4}\log(\sec 2x)\cos 2x + \frac{x}{2}\sin 2x$$

$$y = CF + PI$$

$$= A\cos 2x + B\sin 2x - \frac{1}{4}\log(\sec 2x)\cos 2x + \frac{x}{2}\sin 2x$$

2) $(D^2+4)y = \tan 2x$

AE is $m^2+4=0$ $m^2=-4$ $m=\pm 2i$

CF is $e^{0x}(A\cos 2x + B\sin 2x)$.

$CF = A\cos 2x + B\sin 2x$.

$f_1 = \cos 2x$ $f_2 = \sin 2x$

$f_1' = -2\sin 2x$ $f_2' = 2\cos 2x$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x$$

$$= 2\left[\cos^2 2x + \sin^2 2x\right]$$

$$\boxed{W = 2}$$

$$PI = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2 X}{W}\,dx = -\int \frac{\sin 2x \tan 2x}{2}\,dx = \frac{-1}{2}\int \sin 2x \frac{\sin 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx = \frac{-1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\int \frac{1}{\cos 2x}\,dx + \frac{1}{2}\int \frac{\cos^2 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\int \sec 2x\,dx + \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{-1}{2}\,\frac{\log(\sec 2x + \tan 2x)}{2} + \frac{1}{2}\,\frac{\sin 2x}{2}$$

$$= \frac{-1}{4}\log(\sec 2x + \tan 2x) + \frac{1}{4}\sin 2x.$$

$$Q = \int \frac{f_1 X}{W}\,dx = \int \frac{\cos 2x \tan 2x}{2}\,dx$$

$$= \frac{1}{2}\int \cancel{\cos 2x} \cdot \frac{\sin 2x}{\cancel{\cos 2x}}\,dx = \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{-\cos 2x}{4}.$$

$PI = Pf_1 + Qf_2$.

$= \frac{-1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]\cos 2x - \frac{\cos 2x}{4}\sin 2x$

$y = CF + PI$

$= A\cos 2x + B\sin 2x - \frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]\cos 2x - \frac{\cos 2x \sin 2x}{4}$

3) $(D^2+1)y = \operatorname{cosec} x$

$AE \Rightarrow m^2+1=0 \quad m = \pm i$

$CF = (A\cos x + B\sin x)$

$f_1 = \cos x \qquad f_2 = \sin x$

$f_1' = -\sin x \qquad f_2' = \cos x$

$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x = 1$

$\boxed{W = 1}$

$PI = Pf_1 + Qf_2$

$P = -\int \frac{f_2 X}{W} dx \qquad Q = \int \frac{f_1 X}{W} dx$

$P = -\int \frac{\sin x \operatorname{cosec} x}{1} dx = -\int \cancel{\sin x} \cdot \frac{1}{\cancel{\sin x}} dx$

$\boxed{P = -x}$

$Q = \int \frac{f_1 X}{W} dx = \int \frac{\cos x \operatorname{cosec} x}{1} dx$

$= \int \cos x \cdot \frac{1}{\sin x} dx = \int \cot x \, dx$

$= \frac{\log(\sin x)}{1} \Rightarrow \boxed{Q = \log(\sin x)}$

$PI = -x \cos x + \log(\sin x) \sin x$

$y = CF + PI$

$y = A\cos x + B\sin x - x\cos x + \log(\sin x) \sin x.$

3. $(D^2+1)y = \cot x.$

$AE \Rightarrow m^2+1=0 \Rightarrow m^2 = -1$

$m = \pm i$

$CF \Rightarrow A\cos x + B\sin x$

$f_1 = \cos x \; ; \; f_2 = \sin x \; ; \; f_1' = -\sin x \; ; \; f_2' = \cos x$

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1$$

$$PI = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2 X}{W}\,dx \qquad Q = \int \frac{f_1 X}{W}\,dx$$

$$P = -\int \frac{\sin x \cdot \cot x}{1}\,dx = -\int \sin x \cdot \frac{\cos x}{\sin x}\,dx$$

$$= -\int \cos x\,dx \qquad \boxed{P = -\sin x}$$

$$Q = \int \frac{f_1 X}{W}\,dx = \int \frac{\cos x \cot x}{1}\,dx$$

$$= \int \cos x \cdot \frac{\cos x}{\sin x}\,dx = \int \frac{\cos^2 x}{\sin x}\,dx$$

$$= \int \frac{(1-\sin^2 x)}{\sin x}\,dx = \int \frac{1}{\sin x}\,dx - \int \frac{\sin^2 x}{\sin x}\,dx$$

$$= \int \operatorname{cosec} x\,dx - \int \sin x\,dx$$

$$= -\log(\operatorname{cosec} x + \cot x) - (-\cos x)$$

$$Q = -\log(\operatorname{cosec} x + \cot x) + \cos x$$

$y =$

$$PI = Pf_1 + Qf_2$$

$$= -\sin x \cos x + \left[-\log(\operatorname{cosec} x + \cot x) + \cos x\right] \sin x$$

$$y = CF + PI$$

$$= A\cos x + B\sin x - \sin x \cos x + \left[-\log(\operatorname{cosec} x + \cot x) + \cos x\right] \sin x .$$