Homogeneous Linear ODE'S with constant Coefficients

Let $(a_0D^2+a_1D+a_2)y = \phi(x)$ be the differential equation.

★ The auxillary equation is given by

$$a_0m^2+a_1m+a_2=0$$

This equation has two roots, say m_1 & m_2

★ The complementary function is given by

i) m_1 & m_2 are real and different

$$Ae^{m_1x}+Be^{m_2x}$$

ii) m_1 & m_2 are real and equal

$$(A+Bx)e^{mx}$$

(iii) m_1 & m_2 are complex numbers, say $\alpha+i\beta$

$$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

★ Find particular Integral

★ $y = CF + PI$

Type-I

Solve : $(D^2+3D+2)y = e^{-3x}$ → RHS is in exponential form so (Type I)

AE $m^2+3m+2=0$

$m=-1,-2$

Roots are Real and different

$$CF = Ae^{-x}+Be^{-2x}$$

Particular Integral : $\frac{1}{D^2+3D+2}e^{-3x}$. $D=-3$

Sub $D = -3$

$$= \frac{1}{(-3)^2 + 3(-3) + 2} e^{-3x} = \frac{1}{9-9+2} e^{-3x}$$

$$= \frac{1}{2} e^{-3x}$$

$\Rightarrow y = CF + PI$

$$= Ae^{-x} + Be^{-2x} + \frac{1}{2} e^{-3x}.$$

2. $(D^2 + 4D + 4)y = 11e^{-2x}$.

AE $\Rightarrow$ $m^2 + 4m + 4 = 0$.

$(m+2)(m+2) = 0 \Rightarrow m = -2, -2$

Roots are real and equal.

$$CF = (A + Bx)e^{-2x}$$

Particular Integral :- $\frac{1}{D^2+4D+4} 11e^{-2x}$

$\boxed{D = -2}$

$$= \frac{1}{(-2)^2 + 4(-2) + 4} 11e^{-2x} = \frac{1}{4+4-8} 11e^{-2x}$$

$$= \frac{11e^{-2x}}{0}$$

Diff & multiply by x.

$$= \frac{x}{2D+4} 11e^{-2x} = \frac{x}{2(-2)+4} 11e^{-2x}$$

$$= \frac{x}{0} 11e^{-2x}$$

$$= \frac{x^2}{2} 11e^{-2x}$$

$y = CF + PI$

$$= (A + Bx)e^{-2x} + \frac{x^2}{2} 11e^{-2x}.$$

Type 2 RHS $\sin(ax+b)$ (or) $\cos(ax+b)$

1. Solve $(D^2-3D+2)y = \sin 3x$

AE $\Rightarrow$ $m^2-3m+2=0$

$m = 2, 1$

The roots are real and different

$CF = Ae^{x}+Be^{2x}$

Particular Integral:- $= \frac{1}{D^2-3D+2}\sin 3x$.

$D^2 \to -(a^2)$
$-(3^2)$
-9

$= \frac{1}{-9-3D+2}\sin 3x = \frac{1}{-7-3D}\sin 3x$

$= \frac{1}{-3D-7}\sin 3x = \frac{1}{-3D-7}\times\frac{(-3D+7)}{(-3D+7)}\sin 3x$

$= \frac{-3D+7}{(-3D)^2-(7)^2}\sin 3x = \frac{-3D+7}{+9D^2-49}\sin 3x$

$= \frac{-3D+7}{9(-9)-49}\sin 3x$

$= \frac{-3D+7}{-81-49}\sin 3x = \frac{-3D+7}{-130}\sin 3x$

$= \frac{-1}{130}\left[-3D(\sin 3x)+7\sin 3x\right]$

$= \frac{-1}{130}\left[-3\cos 3x\,(3)+7\sin 3x\right]$

$= \frac{-1}{130}\left[-9\cos 3x+7\sin 3x\right]$

$y = CF+PI$

$y = Ae^{x}+Be^{2x}+\left(\frac{-1}{130}\right)\left[-9\cos 3x+7\sin 3x\right]$

(or) $\frac{1}{130}\left[9\cos 3x-7\sin 3x\right]$

2) $(D^2-3D+2)y = 2\cos(2x+3) + 2e^{x}$

HW

$AE \Rightarrow \quad m^2-3m+2=0$

$m = 2, 1.$

$CF = Ae^{x} + Be^{2x}$

Particular Integral: $PI_1 = \dfrac{1}{D^2-3D+2}\, 2\cos(2x+3)$

$D^2 \to -(a^2) = -(2^2) = -4$

$= 2\dfrac{1}{-4-3D+2} \cos(2x+3)$

$= 2 \cdot \dfrac{1}{-3D-2} \cos(2x+3)$

$= 2\left[\dfrac{1}{-3D-2} \cdot \dfrac{(-3D+2)}{(-3D+2)}\right]\cos(2x+3)$

$= 2\left[\dfrac{-3D+2}{9D^2-4}\right]\cos(2x+3)$

$= 2\left[\dfrac{-3D+2}{9(-4)-4}\right]\cos(2x+3) = \cancel{2}\left[\dfrac{-3D+2}{\underset{20}{\cancel{-40}}}\right]\cos(2$

$= \dfrac{-1}{20}\left[-3D+2\right]\cos(2x+3)$

$= \dfrac{-1}{20}\left[-3D(\cos 2x+3) + 2\cos(2x+3)\right]$

$= \dfrac{-1}{20}\left[3\sin(2x+3)(2) + 2\cos(2x+3)\right]$

$PI_1 = \dfrac{-1}{20}\left[6\sin(2x+3) + 2\cos(2x+3)\right]$

$PI_2 = \dfrac{1}{D^2-3D+2}\, 2e^{x}$

$= \dfrac{1}{1-3(1)+2}\, 2e^{x} = \dfrac{x}{2D-3}\, 2e^{x} = \dfrac{x}{-1}\, 2e^{x}$

$= -2xe$

$y = CF + PI_1 + PI_2$

$= Ae^{x} + Be^{2x} + \left(\frac{-1}{20}\right)[6\sin(2x+3) + 2\cos(2x+3)] - 2xe^{x}$

3. $(D^2+1)y = \sin^2 x$.

HW

$(D^2+1)y = \frac{1-\cos 2x}{2} = \frac{1}{2}e^{0x} - \frac{\cos 2x}{2}$

AE $\Rightarrow$ $m^2+1=0 \Rightarrow m^2=-1 \quad m = \pm i \qquad 0 \pm i$

$\alpha = 0 \quad \beta = 1$

$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

The roots are imaginary

$CF = e^{0x}(A\cos x + B\sin x)$

Particular Integral:

$PI_1 = \frac{1}{D^2+1} \cdot \frac{1}{2}e^{0x} = \frac{1}{(0)^2+1} \cdot \frac{1}{2}e^{0x} = \frac{1}{2}$.

$PI_2 = \frac{1}{D^2+1} \cdot \left(-\frac{\cos 2x}{2}\right)$

$D^2 \rightarrow -(a^2)$
$-(2^2)$
-4

$= \frac{1}{\underset{3}{-4+1}}\left(\frac{-\cos 2x}{2}\right) = \frac{\cos 2x}{6}$

$y = CF + PI$

$= (A\cos x + B\sin x) + \frac{1}{2} + \frac{\cos 2x}{6}$

4. $(D^2+4D+2) = \sin 3x$.

HW

AE $\Rightarrow$ $m^2+4m+2 = 0$

$\Rightarrow \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\Rightarrow \frac{-4 \pm \sqrt{16-4(1)(2)}}{2(1)}$

$\Rightarrow \frac{-4 \pm \sqrt{16-8}}{2}$

$\Rightarrow \frac{-4}{2} \pm \frac{\sqrt{8}}{2} \Rightarrow -2 \pm \sqrt{2}$

$m = -2 \pm \sqrt{2}$

$CF = Ae^{(-2+\sqrt{2})x} + Be^{(-2-\sqrt{2})x}$

Particular Integral: $= \frac{1}{D^2+4D+2}\sin 3x$

$D^2 \rightarrow -(a^2)$
$-(3^2)$
-9

$$= \frac{1}{-9+4D+2}\sin 3x = \frac{1}{-7+4D}\sin 3x$$

$$= \frac{1}{4D-7}\cdot\frac{(4D+7)}{(4D+7)}\sin 3x = \frac{(4D+7)\sin 3x}{16D^2-49}$$

$$= \frac{[4D\sin 3x + 7\sin 3x]}{16(-9)-49} = \frac{4\cos 3x\,(3) + 7\sin 3x}{144-49}$$

$$= \frac{-1}{193}\left[12\cos 3x + 7\sin 3x\right]$$

$y = CF + PI$

$$= Ae^{(-2+\sqrt{2})x} + Be^{(-2-\sqrt{2})x} - \frac{1}{193}\left[12\cos 3x + 7\sin 3x\right]$$

5. $(D^2+4)y = \cos 2x$.

HW $AE \Rightarrow m^2+4=0 \quad m^2=-4 \quad m=\pm 2i$

Roots are imaginary $\Rightarrow CF = e^{0x}\left[A\cos 2x + B\sin 2x\right]$

$\Rightarrow CF = A\cos 2x + B\sin 2x$

$$PI \Rightarrow \frac{1}{D^2+4}\cos 2x = \frac{1}{-4+4}\cos 2x$$

$D^2 \to -(a^2)$

$-(2^2)$

-4

$$= \frac{x\cos 2x}{2D} = \frac{x}{2D}\left(\frac{2D}{2D}\right)\cos 2x$$

$$= \frac{x\,2D\cos 2x}{4D^2} = \frac{-2x\sin 2x\,(2)}{4(-4)}$$

$$= \frac{-4x\sin 2x}{-16} = \frac{x\sin 2x}{4}$$

$y = CF + PI$

$$= A\cos 2x + B\sin 2x + \frac{x\sin 2x}{4}$$

Type-I

3. $(D^2+4D+8)y = e^{2x}+4$

HW

AE $\Rightarrow$ $m^2+4m+8=0$

$m = -2\pm 2i \quad \alpha=-2 \quad \beta=2$

$m = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-4\pm\sqrt{16-4(1)(8)}}{2(1)} = \frac{-4\pm\sqrt{16-32}}{2} = \frac{-4\pm\sqrt{-16}}{2} = \frac{-4\pm 4i}{2} = -2\pm 2i$

$\therefore CF = e^{-2x}(A\cos 2x + B\sin 2x)$

$PI_1 = \frac{1}{D^2+4D+8}e^{2x} = \frac{1}{4+4(2)+8}e^{2x}$

$= \frac{1}{4+8+8}e^{2x} = \frac{1}{20}e^{2x}$

$PI_2 = \frac{1}{D^2+4D+8}4e^{0x} = \frac{1}{0+4(0)+8}4e^{0x}$

$= \frac{4}{8} = 1/2$

$\Rightarrow y = CF+PI$

$= e^{-2x}(A\cos 2x + B\sin 2x) + \frac{1}{20}e^{2x} + \frac{1}{2}.$

4. $(4D^2-4D+1)y=4$

HW AE $\Rightarrow$ $4m^2-4m+1=0$

$m = 1/2, 1/2$

$m = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{4\pm\sqrt{16-4(4)(1)}}{2(4)} = \frac{4\pm\sqrt{0}}{2}$

$= 4/8 = 1/2$

$m = 1/2, 1/2$

$\Rightarrow CF = (A+Bx)e^{1/2x}$

$PI = \frac{1}{4D^2-4D+1}4e^{0x} = \frac{1}{4(0)^2-4(0)+1}4(1) = 4$

$y = CF+PI$

$= (A+Bx)e^{1/2x}+4.$

5) $(D^2 - 7D + 12)y = e^{5x}$

HW

AE $\to$ $m^2 - 7m + 12 = 0$.

$m = 4, 3$

The roots are real and different

$\therefore CF = Ae^{3x} + Be^{4x}$

$$PI = \frac{1}{D^2 - 7D + 12} e^{5x} = \frac{1}{(5)^2 - 7(5) + 12} e^{5x} = \frac{1}{25 - 35 + 12} e^{5x}$$

$$= \frac{1}{2} e^{5x}$$

$y = CF + PI$

$$= Ae^{3x} + Be^{4x} + \frac{1}{2} e^{5x}$$

Type - II

6. $(D^2 + 2D + 1) = \sin 2x \cos 2x$

AE $\Rightarrow$ $m^2 + 2m + 1 = 0 \Rightarrow (m+1)(m+1) = 0$

$m = -1, -1$.

Roots are Real and equal

$\Rightarrow CF = (A + Bx)e^{-x}$

$\sin 2x = 2\sin x \cos x$

$\sin x \cos x = \frac{\sin 2x}{2}$

$x = 2x$

$\sin 2x \cos 2x = \frac{\sin 4x}{2}$

$D^2 + 2D + 1 = \frac{\sin 4x}{2}$

$$PI = \frac{1}{D^2 + 2D + 1} \cdot \left(\frac{\sin 4x}{2}\right) = \frac{1}{-16 + 2D + 1} \cdot \left(\frac{\sin 4x}{2}\right) = \frac{1}{2D - 15} \left(\frac{\sin 4x}{2}\right)$$

$$= \frac{1}{2D - 15} \cdot \left(\frac{2D + 15}{2D + 15}\right)\left(\frac{\sin 4x}{2}\right) = \frac{2D + 15}{4D^2 - 225} \left(\frac{\sin 4x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{2D\sin 4x + 15\sin 4x}{4(-16) - 225}\right] = \frac{1}{2}\left[\frac{2\cos 4x(4) + 15\sin 4x}{-289}\right]$$

$$= -\left[\frac{8\cos 4x + 15\sin 4x}{578}\right]$$

$$y = CF + PI = (A + Bx)e^{-x} - \frac{8\cos 4x + 15\sin 4x}{578}$$