A medium through which heat is conducted may involve the conversion of mechanical, electrical, nuclear, or chemical energy into heat (or thermal energy).

In heat conduction analysis, such conversion processes are characterized as heat generation.

Following are some of the cases where heat generation and heat conduction are encountered :

(*i*) Fuel rods – nuclear reactor;
(*ii*) Electrical conductors;
(*iii*) Chemical and combustion processes;
(*iv*) Drying and setting of concrete.

It is of paramount importance that the heat generation rate be controlled otherwise the equipment may fail (*e.g.*, some nuclear accidents, electrical fuses blowing out). Thus, in the design of the thermal systems temperature distribution within the medium and the rate of heat dissipation to the surroundings assumes ample importance / significance.

For example, the temperature of a resistance wire rises rapidly when electric current passes through it as a result of the electrical energy being converted to heat at a rate of *I*2*R*, where *I* is the current and *R* is the electrical resistance of the wire The safe and effective removal of this heat away from the sites of heat generation (the electronic circuits) is the subject of *electronics cooling,* which is one of the modern application areas of heat transfer.

Likewise, a large amount of heat is generated in the fuel elements of nuclear reactors as a result of nuclear fission that serves as the *heat source* for the nuclear power plants. The natural disintegration of radioactive elements in nuclear waste or other radioactive material also results in the generation of heat throughout the body. The heat generated in the sun as a result of the fusion of hydrogen into helium makes the sun a large nuclear reactor that supplies heat to the earth.

Another source of heat generation in a medium is exothermic chemical reactions that may occur throughout the medium. The chemical reaction in this case serves as a *heat source* for the medium. In the case of endothermic reactions, however, heat is absorbed instead of being released during reaction, and thus the chemical reaction serves as a *heat sink*.

The rate of heat generation in a medium may vary with time as well as position within the medium. When the variation of heat generation with position is known, the *total* rate of heat generation in a medium of volume V can be determined from In the special case of *uniform* heat generation, as in the case of electric resistance heating throughout a homogeneous material, the relation in

In the preceding section we considered conduction problems for which the temperature distribution in a medium was determined solely by conditions at the boundaries of the medium. We now want to consider the additional effect on the temperature distribution of processes that may be occurring *within* the medium. In particular, we wish to consider situations for which thermal energy is being *generated* due to *conversion* from some other energy form.

Refer to Fig. 2.91. Consider a plane wall of thickness L (small in comparison with other dimension) of uniform thermal conductivity k and in which heat sources are uniformly distributed in the whole volume. Let the wall surfaces are maintained at temperatures t_1 and t_2.

Let us assume that heat flow is one-dimensional, under steady state conditions, and there is a *uniform volumetric heat generation* within the wall.

Consider an element of thickness at a distance x from the left hand face of the wall.

Heat conducted in at distance x,

$$Q_x = -kA\frac{dt}{dx}$$

Heat generated in the element,

$$Q_g = A \, . \, dx \, . \, q_g$$

(where q_g = heat generated per unit volume per unit time in the element)

Heat conducted out at distance $(x + dx)$, $Q_{(x+dx)} = Q_x + \frac{d}{dx}(Q_x)dx$

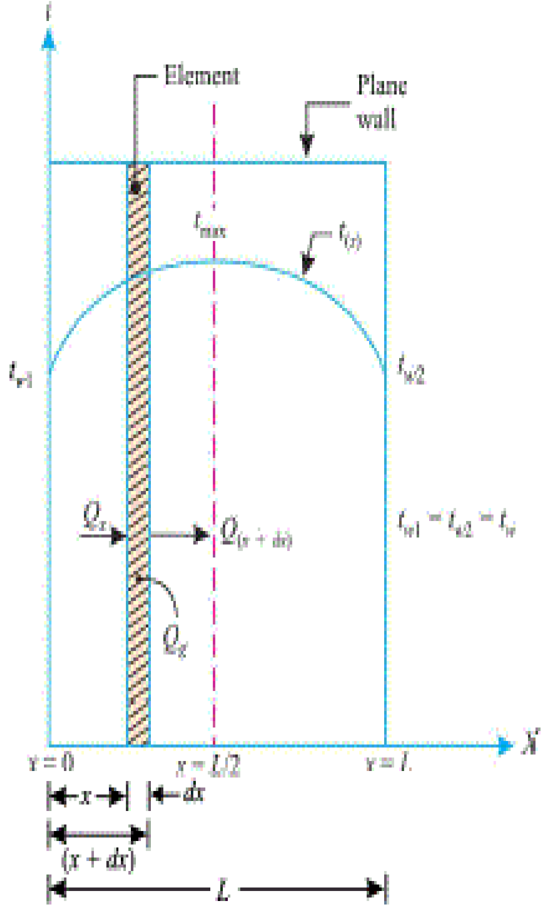

Fig. 2.91. Plane wall uniform heat generation. Both the surfaces maintained at a common temperature.

As Q_g represents an energy increase in the volume element, an energy balance on the element of thick dx is given by

$$Q_x + Q_g = Q_{(x+dx)}$$

$$= Q_x + \frac{d}{dx}(Q_x)\,dx$$

or, $$Q_g = \frac{d}{dx}(Q_x)\,dx$$

or, $$q_g . A . dx = \frac{d}{dx}\left[-k\,A\frac{dt}{dx}\right]dx$$

$$= -k\,A . \frac{d^2t}{dx^2} . dx$$

or, $$\frac{d^2t}{dx^2} + \frac{q_g}{k} = 0 \qquad ...(2.88)$$

Eqn. (2.88) may also be obtained from eqn. (2.8) by assuming one-dimensional steady state conditions.

The first and second integration of Eqn. (2.88) gives respectively

$$\frac{dt}{dx} = -\frac{q_g}{k}x + C_1 \qquad ...(2.89)$$

$$t = -\frac{q_g}{2k}\cdot x^2 + C_1\,x + C_2 \qquad ...\ (2.90)$$

Case I. Both the surfaces have the same temperature :

Refer to Fig. 2.92.

At $x = 0$ $\quad t = t_1 = t_w$, and

At $x = L$ $\quad t = t_2 = t_w$

(where t_w = temperature of the wall surface).

Using these boundary conditions in eqn. (2.90), we get

$$C_2 = t_w \text{ and } C_1 = \frac{q_g}{2k}\cdot L$$

Substituting these values of C_1 and C_2 in eqn. (2.90), we have

$$t = -\frac{q_g}{2k}x^2 + \frac{q_g}{2k}\cdot L\cdot x + t_w$$

or, $$t = \frac{q_g}{2k}(L - x)\,x + t_w \qquad ...(2.91)$$

In order to determine the location of the maximum temperature, differentiating the eqn. (2.91) *w.r.t x* and equating the derivative to *zero*, we have

$$\frac{dt}{dx} = \frac{q_g}{2k}(L - 2x) = 0$$

Since, $\frac{q_g}{2k} \neq 0$, therefore,

$$L - 2x = 0 \qquad \text{or} \qquad x = \frac{L}{2}$$

Thus the *distribution of temperature* given by eqn. (2.91) is the *parabolic* and *symmetrical* about the midplane. The maximum temperature occurs at $x = \frac{L}{2}$ and its value equals

$$t_{max} = \left[\frac{q_g}{2k}(L - x)\,x\right]_{x=\frac{L}{2}} + t_w$$

or,
$$= \left[\frac{q_g}{2k}\left(L - \frac{L}{2}\right)\frac{L}{2}\right] + t_w$$

i.e.
$$t_{max} = \frac{q_g}{8k} \cdot L^2 + t_w \qquad ...(2.92)$$

Heat transfer then takes place towards both the surfaces, and for each surface it is given by

$$Q = -kA\left(\frac{dt}{dx}\right)_{x=0 \text{ or } x=L}$$
$$= -kA\left[\frac{q_g}{2k}(L - 2x)\right]_{x=0 \text{ or } x=L}$$

i.e.,
$$Q = \frac{AL}{2} \cdot q_g \qquad ...(2.93)$$

When both the surfaces are considered,

$$Q = 2 \times \frac{AL}{2} q_g = A.L.q_g \qquad ...[2.93\ (a)]$$

Also heat conducted to *each wall surface* is further dissipated to the surrounding atmosphere at temperature t_a.

Thus,
$$\frac{AL}{2} \cdot q_g = hA(t_w - t_a)$$

or,
$$t_w = t_a + \frac{q_g}{2h} \cdot L \qquad ...(2.94)$$

Substituting this value of t_w in eqn. (2.91), we obtain

$$t = t_a + \frac{q_g}{2h} \cdot L + \frac{q_g}{2k}(L - x)x \qquad ...(2.95)$$

At $x = L/2$ *i.e.*, at the midplane :

$$t = t_{max} = t_a + \frac{q_g}{2h} . L + \frac{q_g L^2}{8k}$$

or,
$$t_{max} = t_a + q_g\left[\frac{L}{2h} + \frac{L^2}{8k}\right] \qquad ...[2.95\ (a)]$$

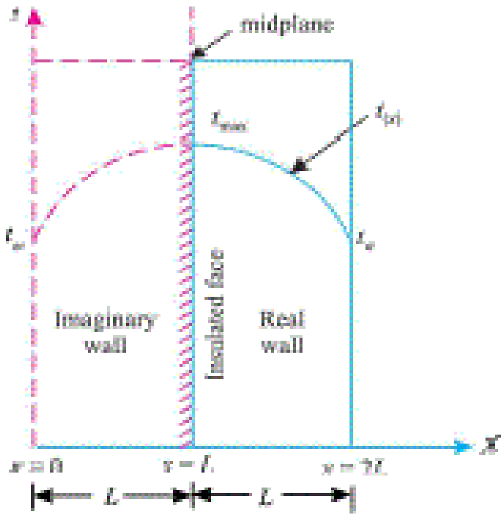

Fig. 2.92. Heat conduction in an insulated wall.

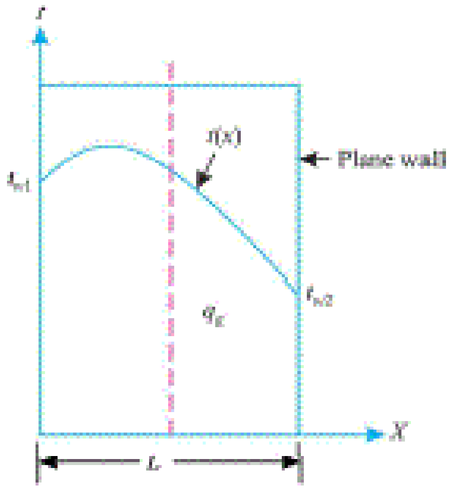

Fig. 2.93. Plane wall with uniform heat generation—Both the surfaces of the wall having different temperatures.

The eqn. (2.95) also works well in case of conduction in an *insulated wall* Fig. (2.92).

The following boundary conditions apply in the full *hypothetical wall* of thickness 2L :

At $x = L$ $\quad \frac{dt}{dx} = 0$

At $x = 2L$ $\quad t = t_w$

The location $x = L$ refers to the mid-plane of the hypothetical wall (or insulated face of given wall).

Eqns. (2.91) and (2.92) for temperature distribution and maximum temperature at the mid- plane (insulated end of the given wall) respectively can be written as

$$t = \frac{q_g}{2k}(2L - x)\,x + t_w \quad ...(2.96)$$

$$t_{max} = \frac{q_g}{2k}L^2 + t_w \quad ...(2.97)$$

[Substituting L = 2L in eqn. (2.91) and (2.92)]

Vertical tank.

Case II. Both the surfaces of the wall have different temperatures :

Refer to Fig. 2.93

The boundary conditions are :

At $x = 0$ $\qquad t = t_{w1}$

At $x = L$ $\qquad t = t_{w2}$

Substituting these values in eqn. (2.90), we obtain the values of constant C_1 and C_2 as :

$$C_2 = t_{w1}; \qquad C_1 = \frac{t_{w2} - t_{w1}}{L} + \frac{q_g}{2k}\cdot L$$

Inserting these values in eqn. (2.90), we get

$$t = -\frac{q_g}{2k}x^2 + \frac{t_{w2} - t_{w1}}{L}x + \frac{q_g}{2k}L.x + t_{w1}$$

$$= \frac{q_g}{2k}L\cdot x - \frac{q_g}{2k}x^2 + \frac{x}{L}(t_{w2} - t_{w1}) + t_1$$

or, $$t = \left[\frac{q_g}{2k}(L - x) + \frac{t_{w2} - t_{w1}}{L}\right]x + t_{w1} \quad ...(2.98)$$

The temperature distribution, in dimensionless form can be obtained by making the following transformations :

$$t - t_{w2} = \frac{q_g}{2k}L^2\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right] + \frac{x}{L}(t_{w2} - t_{w1}) + (t_{w1} - t_{w2})$$

or, $$\frac{t - t_{w2}}{t_{w1} - t_{w2}} = \frac{q_g}{2k}\cdot\frac{L^2}{(t_{w1} - t_{w2})}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right] - \frac{x}{L} + 1$$

or, $$\frac{t - t_{w2}}{t_{w1} - t_{w2}} = \frac{q_g}{2k}\cdot\frac{L^2}{(t_{w1} - t_{w2})}\cdot\frac{x}{L}\left[1 - \frac{x}{L}\right] + \left[1 - \frac{x}{L}\right]$$

Replacing the parameter $\frac{q_g}{2k}\cdot\frac{L^2}{(t_{w1} - t_{w2})}$ (a constant) by a factor Z, we have

$$\frac{t - t_{w2}}{t_{w1} - t_{w2}} = Z.\frac{x}{L}\left[1 - \frac{x}{L}\right] + \left[1 - \frac{x}{L}\right]$$

or, $$\frac{t - t_{w2}}{t_{w1} - t_{w2}} = \left[1 - \frac{x}{L}\right]\left[\frac{Z\,x}{L} + 1\right] \quad ...(2.99)$$

In order to get maximum temperature and its location, differentiating Eqn. (2.99) w.r.t x and equating the derivative to zero, we have

$$\frac{dt}{d(x/L)} = \left(1 - \frac{x}{L}\right)Z + \left(\frac{Zx}{L} + 1\right)(-1) = 0$$

or, $$Z - \frac{Zx}{L} - \frac{Zx}{L} - 1 = 0$$

or, $$\frac{2Zx}{L} = Z - 1$$

or, $$\frac{x}{L} = \frac{Z - 1}{2Z} \quad ...(2.100)$$

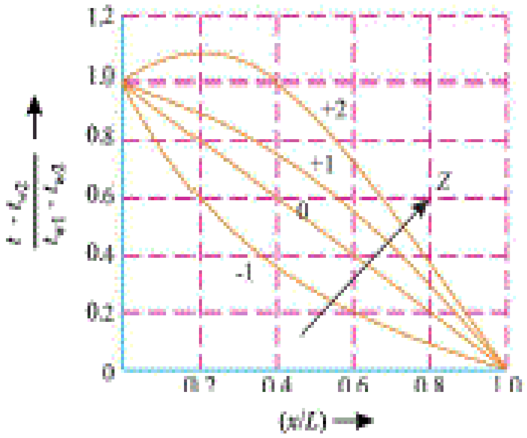

Fig. 2.94. Effect of factor Z on the temperature distribution in the plane wall.

Thus the maximum value of temperature occurs at $\frac{x}{L} = \frac{Z-1}{2Z}$ and its value is given by:

$$\frac{t_{max} - t_{w2}}{t_{w1} - t_{w2}} = \left[1 - \frac{Z-1}{2Z}\right]\left[Z \times \left(\frac{Z-1}{2Z}\right) + 1\right]$$

or, $$\frac{t_{max} - t_{w2}}{t_{w1} - t_{w2}} = \left(\frac{Z+1}{2Z}\right)\left(\frac{Z+1}{2}\right)$$

$$= \frac{(Z+1)^2}{4Z} \quad ...(2.101)$$

Fig. 2.94 shows the effect of factor Z on the temperature distribution in the plane wall. The following points emerge :

- As the value of Z *increases* the slope of the curve changes; obviously the direction of heat flow can be reversed by an adequately large value of q_g.
- When $Z = 0$, the temperature distribution is *linear* (*i.e.*, no internal heat generation).
- When the value of Z is *negative*, q_g represents *absorption of heat* within the wall/body.

Case III. Current carrying electrical conductor :

When electrical current passes through a conductor, heat is generated (Q_g) in it and is given by

$$Q_g = I^2R, \text{ where } R = \frac{\rho L}{A}$$

where,
I = Current flowing in the conductor,
R = Electrical resistance,
ρ = Specific resistance or resistivity,
L = Length of the conductor, and
A = Area of cross-section of the conductor.

Also, $$Q_g = q_g \times A \times L$$

$$\therefore \quad q_g \times A \times L = I^2 \times \frac{\rho L}{A} \quad \text{or,} \quad q_g = I^2 \times \frac{\rho L}{A} \times \frac{1}{AL} = \frac{I^2\rho}{A^2}$$