Topic - One Dimensional Steady State Heat Conduction through Plane Wall

Conduction heat transfer:

Objectives of conduction analysis:

1) Primary objective is to determine the temperature field $T(x)$ in a body [i.e how temperature varies with position within the body].

2) $T(x)$ depends on boundary condition, initial condition, material properties (ρ, k, C_p), and geometry.

3) Why we need temperature $T(x)$.

a) Compute heat flux at any point (using Fourier eqn).

b) Compute thermal stresses, expansion, deflection due to temp etc,

c) Design products in application such as insulation thickness, chip temperature calculations (electronics), Heat treatment of metals.

Boundary and Initial conditions:

1) Heat equation is second order in spatial coordinate. Hence, 2 BC's needed for each coordinate.

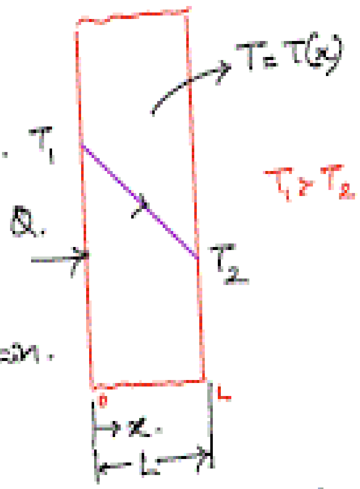
* 1D problem : 2 BC in x-direction.

2) Heat equation is first order in time. Hence 1 IC is needed.

One-dimensional steady state heat conduction without heat generation: $q'' = 0$, $\frac{\partial E}{\partial t} = 0$.

$$\frac{\partial}{\partial x}\left(k.\frac{\partial T}{\partial x}\right) = 0. \rightarrow k.\frac{d^2T}{dx^2} = 0. \rightarrow (1)$$

$$\frac{d^2T}{dx^2} = 0.$$



Integrating the equation once.

$$\frac{dT}{dx} = C_1 \rightarrow (2)$$

Integrating the equation again.

$$T = C_1 x + C_2 \rightarrow (3)$$

To determine the constants C_1 & C_2, apply the boundary conditions:

i.e @ $x = 0$, $T = T_1$ $\rightarrow$ $T_1 = C_1 0 + C_2$ $\rightarrow$ $C_2 = T_1 \rightarrow (4)$

& @ $x = L$, $T = T_2$. $\rightarrow$ $T_2 = C_1 L + C_2$

$$T_2 = C_1 L + T_1 \quad \because C_2 = T_1$$

$$\therefore C_1 = \frac{(T_2 - T_1)}{L} \rightarrow (5)$$

$\therefore$ (3) becomes,

$$T = \frac{(T_2 - T_1)}{L} x + T_1 \rightarrow \frac{T - T_1}{T_2 - T_1} = \frac{x}{L} \rightarrow (6)$$

$T(x)$.

Temperature distribution across the plane wall.

Rate of heat transfer through the plane wall:

$$Q = -KA\frac{dT}{dx} \rightarrow (7)$$ [Fourier conduction equation]

$$Q = -K.A\frac{T_2-T_1}{L} \qquad \because C_1 = \frac{T_2-T_1}{L}.$$

$$Q = K.A\frac{(T_1-T_2)}{L}$$

$(T_1-T_2) \rightarrow \Delta T = (T_1-T_2)$

$L \rightarrow$ thickness of the wall.

$$Q = \frac{(T_1-T_2)}{\left[\frac{L}{KA}\right]} \rightarrow (8)$$

Thermal resistance concept using Ohm's law:

$V_1 \xrightarrow{i} R \;—\; V_2$.

$$V = I.R \rightarrow I = \frac{V}{R} \rightarrow (9).$$

Comparing equations (8) and (9), we have.

$$Q = I;\; (T_1-T_2) = V \text{ and } \frac{L}{KA} = R.$$

∴ Thermal resistance for the plane wall.

$$R_{th} = \frac{L}{KA} \rightarrow (10) \qquad \left[\frac{m}{W/mK \cdot m^2} = \frac{m^2 \cdot K}{W \cdot m^2} = \frac{K}{W}\right].$$

Thermal resistance circuit for plane wall.

$T_1 \xrightarrow{Q}$ $R_{th} = \frac{L}{KA}$ T_2 unit $\frac{K}{W}$ (or) $°C/W$.

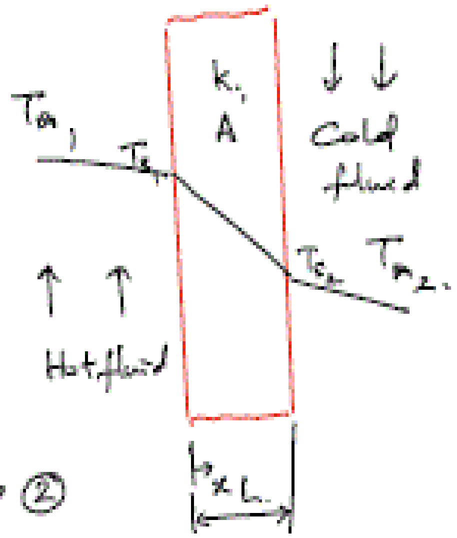
Thermal resistance for convection from newton's law of cooling $Q = hA_s(T_s - T_\infty) \rightarrow Q = \frac{(T_s - T_\infty)}{1/hA_s}$.

$$\therefore R_{th\,conv} = \frac{1}{hA_s}$$

Plane wall with convective boundary conditions:

The differential equation is:

$$\frac{d}{dx}\left(k.\frac{dT}{dx}\right)=0 \rightarrow (1)$$



Integrating twice,

$$T(x) = C_1 x + C_2.$$

Boundary conditions are.

@ $x=0$, $T=T_{s_1}$ $\quad T(0)=T_{s_1} \rightarrow (2)$

@ $x=L$, $T=T_{s_2}$ $\quad T(L)=C_1 L + T_{s_1}$

$$\therefore C_1 = \frac{T(L)-T_{s_1}}{L} \rightarrow (3)$$

$$\therefore T(x) = (T(L)-T_{s_1})\frac{x}{L} + T_{s_1} \rightarrow (4)$$

Heat flux across the wall is given by

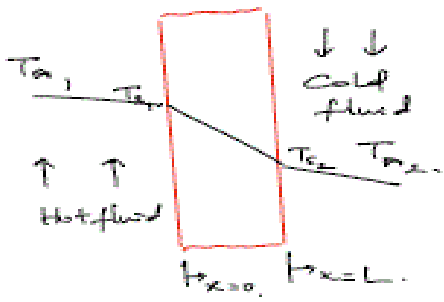
$$q=-kA\frac{dT}{dx} = \frac{k.A}{L}(T_{s_1}-T_{s_2}) = \frac{T_{s_1}-T_{s_2}}{\left(\frac{L}{kA}\right)} \rightarrow (5)$$

Thermal resistance concept for convection:

$$q=hA(T_{\infty_1}-T_{s_1}) \rightarrow q=\frac{(T_{\infty_1}-T_{s_1})}{1/hA}.$$

$$\therefore R_{th_{conv}} = \frac{T_{\infty_1}-T_{s_1}}{q} = \frac{1}{hA}$$

Applying thermal resistance concept for the plane wall with convection boundary conditions.



$$T_{\infty_1} \xrightarrow{q} \frac{1}{h_1 A} \; T_{s_1} \xrightarrow{q} \frac{L}{KA} \; T_{s_2} \xrightarrow{q} \frac{1}{h_2 A} \; T_{\infty_2}.$$

Heat transfer rate may be determined by considering each element of the resistance network, as.

$$q = \frac{T_{\infty_1} - T_{s_1}}{\frac{1}{h_1 A}} = \frac{T_{s_1} - T_{s_2}}{\frac{L}{KA}} = \frac{T_{s_2} - T_{\infty_2}}{\frac{1}{h_2 A}} \longrightarrow (1)$$

Since resistances are in series,

$$R_{total} = \sum R_{th} = \frac{1}{h_1 A} + \frac{L}{KA} + \frac{1}{h_2 A}. \longrightarrow (2)$$

$$= \frac{T_{\infty_1} - T_{s_1}}{q} + \frac{T_{s_1} - T_{s_2}}{q} + \frac{T_{s_2} - T_{\infty_2}}{q}.$$

$$= \frac{T_{\infty_1} - \cancel{T_{s_1}} + \cancel{T_{s_1}} - \cancel{T_{s_2}} + \cancel{T_{s_2}} - T_{\infty_2}}{q}.$$

$$\left.\begin{aligned} R_{total} &= \frac{T_{\infty_1} - T_{\infty_2}}{q}. \\ q &= \frac{T_{\infty_1} - T_{\infty_2}}{R_{total}} \end{aligned}\right\} \longrightarrow (I)$$