2]. Solve $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x+1$

Soln.

Given $[(3x+2)^2D^2+3(3x+2)D-36]y=3x^2+4x+1$ ↳(1)

Take $3x+2=e^z \Rightarrow 3x=e^z-2 \Rightarrow x=\frac{e^z-2}{3}$

$\log(3x+2)=z$

$(3x+2)D=3D'$

$(3x+2)^2D^2=9D'(D'-1)$

(1) $\Rightarrow [9D'(D'-1)+3(3D')-36]y=3\left[\frac{e^z-2}{3}\right]^2+4\left[\frac{e^z-2}{3}\right]+1$

$[9D'^2-9D'+9D'-36]y=\frac{3}{9}[e^{2z}+4-4e^z]+\frac{4}{3}e^z-\frac{8}{3}+1$

$[9D'^2-36]y=\frac{e^{2z}}{3}+\frac{4}{3}-\frac{4e^z}{3}+\frac{4}{3}e^z-\frac{8}{3}+1$

$=\frac{e^{2z}}{3}-\frac{1}{3}$

$9(D'^2-4)y=\frac{1}{3}[e^{2z}-1]$

$(D'^2-4)y=\frac{1}{27}[e^{2z}-1]$

CF: $m^2-4=0 \Rightarrow m^2=4$

$m=\pm 2$

$CF=Ae^{2z}+Be^{-2z}$

$PI=\frac{1}{D^2-4}\left[\frac{1}{27}(e^{2z}-1)\right]$

$=\frac{1}{27}\left[\frac{1}{D^2-4}e^{2z}-\frac{1}{D^2-4}e^{0z}\right]$

$=\frac{1}{27}\left[z\frac{1}{2D'}e^{2z}-\frac{1}{(-4)}e^{0z}\right]$ $1\to D'\to 2$, $2\to D'\to 0$

$=\frac{1}{27}\left[\frac{z}{4}e^{2z}+\frac{1}{4}\right]$

$=\frac{1}{108}[ze^{2z}+1]$

The Soln. is, $y=CF+PI=Ae^{2z}+Be^{-2z}+\frac{1}{108}[ze^{2z}+1]$

$y=A(3x+2)^2+B(3x+2)^{-2}+\frac{1}{108}[\log(3x+2)[3x+2]^2+1]$