



Legendre's linear differential Equation

An eqn. of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a^2 (ax+b)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} (ax+b) \frac{dy}{dx} + a_n y = Q(x) \rightarrow (1)$$

Take  $ax+b = e^z$

$$z = \log(ax+b)$$

$$(ax+b)D = aD'$$

$$(ax+b)^2 D^2 = a^2 D'(D'-1)$$

$$(ax+b)^3 D^3 = a^3 D'(D'-1)(D'-2) \text{ and so on.}$$

1]. Transform the equation to constant coefficients eqn.

$$(2x+3)^2 y'' - (2x+3)y' + 2y = 6x$$

Soln.

$$\text{Given } [(2x+3)^2 D^2 - (2x+3)D + 2]y = 6x$$

$$\text{Take } 2x+3 = e^z \Rightarrow 2x = e^z - 3 \rightarrow (1)$$

$$z = \log(2x+3) \rightarrow x = \frac{e^z - 3}{2}$$

$$(2x+3)D = 2D'$$

$$(2x+3)^2 D^2 = 4D'(D'-1)$$

$$(1) \Rightarrow [4D'(D'-1) - 2D' + 2]y = 6 \left[ \frac{e^z - 3}{2} \right]$$

$$[4D'^2 - 4D' - 2D' + 2]y = 3[e^z - 3]$$

$$[4D'^2 - 6D' + 2]y = 3e^z - 9 \text{ which is a linear eqn. with constant coefficients.}$$

2]. Solve  $(x+2)^2 \frac{d^2 y}{dx^2} - (x+2) \frac{dy}{dx} + y = 3x + 4$



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Soln.

$$\text{Given } [(x+2)^2 D^2 - (x+2)D + 1]y = 3x+4 \rightarrow (1)$$

$$\text{Take } x+2 = e^z \Rightarrow x = e^z - 2 \\ z = \log(x+2)$$

$$(x+2)D = D' \\ (x+2)^2 D^2 = D'(D'-1)$$

$$(1) \Rightarrow [D'(D'-1) - D' + 1]y = 3[e^z - 2] + 4 \\ [D'^2 - D' - D' + 1]y = 3e^z - 6 + 4 \\ [D'^2 - 2D' + 1]y = 3e^z - 2$$

CF

$$m^2 - 2m + 1 = 0 \\ (m-1)(m-1) = 0 \\ m = 1, 1$$

$$CF = (A + Bz)e^z$$

$$PI = \frac{1}{D'^2 - 2D' + 1} 3e^z + \frac{1}{D'^2 - 2D' + 1} 2e^{0z} \\ = \frac{1}{1 - 2 + 1} 3e^z - 2 \frac{1}{1} e^{0z} \quad \begin{matrix} D' \rightarrow a=1 \\ a=0 \end{matrix} \\ = z \frac{1}{2D' - 2} 3e^z - 2 \\ = z \frac{1}{2(1) - 2} 3e^z - 2 \\ = z^2 \frac{1}{2} 3e^z - 2$$

$$PI = \frac{3z^2 e^z}{2} - 2$$

The Soln. is,

$$y = CF + PI$$

$$= (A + Bz)e^z + \frac{3z^2 e^z}{2} - 2$$

$$y = [A + B \log(x+2)](x+2) + \frac{3[\log(x+2)]^2 (x+2)}{2} - 2$$

