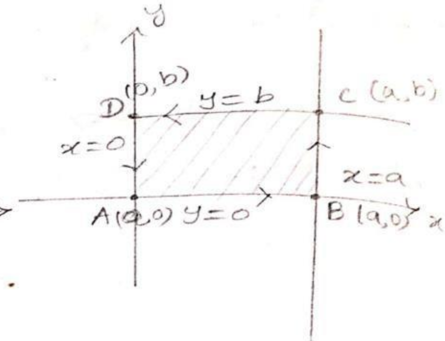
2]. verify Stoke's theorem $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region bounded by $x=0$, $x=a$, $y=0$, $y=b$.

Soln.



ST

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds$$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[2y+2y]$$

$$\nabla \times \vec{F} = 4y\vec{k}$$

RHS

$$\iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_0^a 4y\vec{k} \cdot \vec{k} \, dx \, dy$$

$$= \int_0^b \int_0^a 4y \, dx \, dy$$

$$= 4\int_0^b y [x]_0^a \, dy$$

$$= 4a \int_0^b y \, dy$$

$$= 4a \left[\frac{y^2}{2}\right]_0^b$$

$$\iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds = 2ab^2 \longrightarrow (1)$$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2)dx + 2xy \, dy$$

LHS

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB $[y=0 \Rightarrow dy=0]$

$$\int_{AB} (x^2-y^2)\,dx + 2xy\,dy = \int_0^a x^2\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^a$$

$$= \frac{a^3}{3}$$

Along BC $[x=a \Rightarrow dx=0]$

$$\int_{BC} (x^2-y^2)\,dx + 2xy\,dy = \int_0^b 2ay\,dy$$

$$= 2a\left(\frac{y^2}{2}\right)_0^b$$

$$= ab^2$$

Along CD $[y=b \Rightarrow dy=0]$

$$\int_{CD} (x^2-y^2)\,dx + 2xy\,dy = \int_a^0 (x^2-b^2)\,dx$$

$$= \left[\frac{x^3}{3} - b^2x\right]_a^0$$

$$= 0 - \left(\frac{a^3}{3} - ab^2\right)$$

$$= -\frac{a^3}{3} + ab^2$$

Along DA $[x=0 \Rightarrow dx=0]$

$$\int_{DA} (x^2-y^2)\,dx + 2xy\,dy = 0$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$= 2ab^2 \rightarrow (2)$$

From (1) & (2), LHS = RHS

Hence Stoke's theorem is verified.