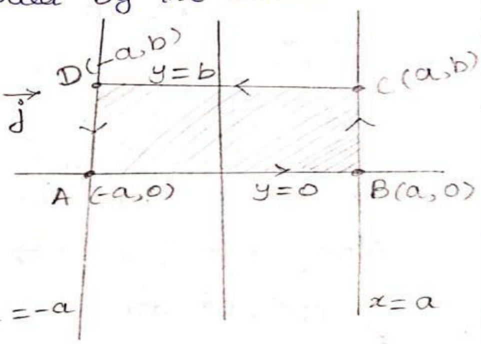
Stoke's Theorem:

The line integral of the tangential component of a vector function $\vec{F}$ around a simple closed curve C is equal to the surface integral of the normal component of curl $\vec{F}$ over an open surface S.

i.e., $$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS$$

1. Verify Stoke's theorem for $\vec{F} = (x^2+y^2)\vec{i} - 2xy\,\vec{j}$ taken around the rectangle bounded by the lines $x=\pm a$, $y=0$, $y=b$.

Soln.



Given $\vec{F} = (x^2+y^2)\vec{i} - 2xy\,\vec{j}$

ST

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$$

Now,

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y-2y]$$

$$= -4y\,\vec{k}$$

RHS

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds = \iint_S (-4y\vec{k})\cdot\vec{k}\, dx\, dy$$

$$= \iint_S (-4y)\, dx\, dy$$

$$= -4\int_0^b\int_{-a}^{a} y\, dx\, dy$$

$$= -4\int_0^b y\,[x]_{-a}^{a}\, dy$$

$$= -4\int_0^b y\,[a+a]\,dy$$

$$= -8a\int_0^b y\,dy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = -4ab^2 \rightarrow (1)$$

Given $\vec{F} = (x^2+y^2)\,\vec{i} - 2xy\,\vec{j}$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F}\cdot d\vec{r} = (x^2+y^2)\,dx - 2xy\,dy$$

LHS:

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB $[y=0 \Rightarrow dy=0]$

$$\int_{AB} (x^2+y^2)\,dx - 2xy\,dy = \int_{-a}^{a} x^2\,dx$$

$$= \left(\frac{x^3}{3}\right)_{-a}^{a}$$

$$= \frac{a^3}{3} - \frac{(-a)^3}{3}$$

$$= \frac{2a^3}{3}$$

$$\int_{AB} (x^2+y^2)\,dx - 2xy\,dy = \frac{2a^3}{3}$$

Along BC $[x=a \Rightarrow dx=0]$

$$\int_{BC} (x^2+y^2)\,dx - 2xy\,dy = \int_0^b [0 - 2ay\,dy]$$

$$= -2a\int_0^b y\,dy$$

$$= -2a\left[\frac{y^2}{2}\right]_0^b$$

$$\int_{BC} (x^2+y^2)dx - 2xy\,dy = -ab^2$$

Along CD $[y = b \Rightarrow dy = 0]$

$$\int_{CD} (x^2+y^2)\,dx - 2xy\,dy = \int_a^{-a} (x^2+b^2)\,dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_a^{-a}$$

$$= \left(-\frac{a^3}{3} - ab^2\right) - \left(\frac{a^3}{3} + ab^2\right)$$

$$= -2ab^2 - \frac{2a^3}{3}$$

Along DA $(x = -a \Rightarrow dx = 0)$

$$\int_{DA} (x^2+y^2)dx - 2xy\,dy = \int_b^0 0 - 2(-a)y\,dy$$

$$= \int_b^0 2ay\,dy$$

$$= 2a\left[\frac{y^2}{2}\right]_b^0$$

$$= 0 - ab^2$$

$$= -ab^2$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

$$= \frac{2a^3}{3} - ab^2 - 2ab^2 - \frac{2a^3}{3} - ab^2$$

$$= -4ab^2 \rightarrow (2)$$

From (1) & (2), LHS = RHS

Hence Stoke's theorem is verified.