## Gauss Divergence theorem :

The surface integral of normal component of vector function F over a closed surface S enclosing volume V is equal to the volume integral of divergence of F taking through out the volume V

i.e $\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$

Verify the gauss divergence theorm (GDT) for $\vec{F} = 4xz\,\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$ over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

$$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y = 4z - y.$$

$$\nabla\cdot\vec{F} = 4z - y.$$

RHS:

$$\iiint_V \nabla\cdot\vec{F}\, dv = \int_0^1\int_0^1\int_0^1 (4z - y)\, dx\, dy\, dz.$$

$$= \int_0^1\int_0^1 (4xz - yx)\Big]_{x=0}^{1} dy\, dz$$

$$= \int_0^1\int_0^1 (4z - y)\, dy\, dz.$$

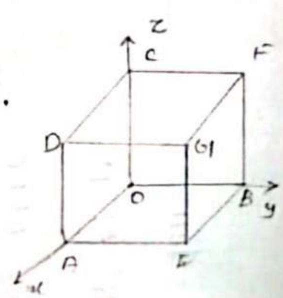
$$= \int_0^1 \left(4zy - \frac{y^2}{2}\right)\Big]_{y=0}^{1} dz.$$

$$= \int_0^1 (4z - 1/2)\, dz.$$

$$= \left[\frac{4z^2}{2} - \frac{1}{2}z\right]_{z=0}^{1}$$

$$= \frac{4}{2} - \frac{1}{2}.$$

$$\iiint_V \nabla\cdot\vec{F}\, dv = 3/2 \quad \text{---} (1).$$



| face | $\hat{n}$ | $\vec{F}\cdot\hat{n}$ | ds | eqn | $\vec{F}\cdot\hat{n}$ on S | $\iint_S \vec{F}\cdot\hat{n}\, ds$ |
|---|---|---|---|---|---|---|
| $S_1$ AEGD | $\vec{i}$ | $4xz$ | $dydz$ | $x=1$ | $4z$ | $\int_0^1\int_0^1 4z\, dydz$. |
| $S_2$ OBFC. | $-\vec{i}$ | $-4xz$. | $dydz$. | $x=0$ | $0$ | $0$ |
| $S_3$ EBFG | $\vec{j}$ | $-y^2$ | $dxdz$. | $y=1$ | $-1$ | $\int_0^1\int_0^1 (-1)\, dxdz$ |
| $S_4$ OADC | $-\vec{j}$ | $+y^2$ | $dxdz$ | $y=0$ | $0$ | $\int 0$ |
| $S_5$ DGFC. | $\vec{k}$ | $yz$ | $dxdy$ | $z=1$ | $y$ | $\int_0^1\int_0^1 y\, dxdy$ |
| $S_6$ OAEB | $-\vec{k}$ | $-yz$ | $dxdy$ | $z=0$ | $0$ | $0$ |

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \ldots + \iint_{S_2} + \iint_{S_3} \ldots + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}.$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 4z\,dy\,dz + 0.$$

$$= \int_0^1 4z(y)_0^1\,dz.$$

$$= \int_0^1 4z\,dz.$$

$$= 4\left(\frac{z^2}{2}\right)_0^1.$$

$$= 4/2.$$

$$= 2.$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds + \iint_{S_4} \vec{F}\cdot\hat{n}\,ds = -\int_0^1\int_0^1 dx\,dz + 0.$$

$$= -\int_0^1 [x]_0^1\,dz.$$

$$= -\int_0^1 dz = -(z)_0^1.$$

$$= -1$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds + \iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 y\,dx\,dy.$$

$$= \int_0^1 y\,[x]_0^1\,dy.$$

$$= \int_0^1 y\,dy.$$

$$= \left[\frac{y^2}{2}\right]_0^1.$$

$$= 1/2.$$

$$\int \vec{F}\cdot\hat{n}\,ds = 2 - 1 + 1/2.$$

$$= 3/2 \quad \text{—— (2)}.$$

from (1) & (2) LHS = RHS Hence verified.

2. Verify gauss divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ where S is the surface of the cuboid formed by the planes $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, $z=c$.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

Now :

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}z^2.$$

$$= 2x + 2y + 2z.$$

$$\nabla\cdot\vec{F} = 2(x+y+z)$$

RHS :

$$\iiint_V \nabla\cdot\vec{F}\,dv$$

$$= \int_0^c\int_0^b\int_0^a 2(x+y+z)\,dx\,dy\,dz.$$

$$= 2\int_0^c\int_0^b \left(\frac{x^2}{2} + yx + zx\right)_{x=0}^{a} dy\,dz.$$

$$= 2\int_0^c\int_0^b \left[\frac{a^2}{2} + ay + az\right] dy\,dz.$$

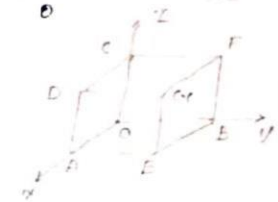
$$= 2\int_0^c \left[\frac{a^2}{2}y + \frac{ay^2}{2} + ayz\right]_{y=0}^{b} dz$$

$$= 2\int_0^c \left[ \frac{a^2 b}{2} + \frac{a b^2}{2} + abz \right] dz$$

$$= 2\left[ \frac{a^2 b}{2} z + \frac{a b^2}{2} z + ab\frac{z^2}{2} \right]_0^c$$

$$= 2\left[ \frac{a^2 bc}{2} + \frac{a b^2 c}{2} + \frac{abc^2}{2} \right]$$

$$= 2\frac{abc}{2}\left[ a+b+c \right]$$



$$\iiint_V \nabla\cdot\vec{F}\, dv = abc\left[ a+b+c \right] \quad \text{—(1)}$$

| Face | $\hat{n}$ | $\vec{F}\cdot\hat{n}$ | eqn | $\vec{F}\cdot\hat{n}$ on S | ds | $\iint \vec{F}\cdot\hat{n}\, ds$ |
|---|---|---|---|---|---|---|
| AEGD | $\vec{i}$ | $x^2$ | $x = a$ | $a^2$ | $dy\,dz$ | $\int_0^c\int_0^b a^2\,dy\,dz$ |
| OBFC | $-\vec{i}$ | $-x^2$ | $x = 0$ | $0$ | $dy\,dz$ | $0$ |
| EBFG | $\vec{j}$ | $y^2$ | $y = b$ | $b^2$ | $dx\,dz$ | $\int_0^c\int_0^a b^2\,dx\,dz$ |
| OADC | $-\vec{j}$ | $-y^2$ | $y = 0$ | $0$ | $dx\,dz$ | $0$ |
| DGFC | $\vec{k}$ | $z^2$ | $z = c$ | $c^2$ | $dx\,dy$ | $\int_0^b\int_0^a c^2\,dx\,dy$ |
| OAEB | $-\vec{k}$ | $-z^2$ | $z = 0$ | $0$ | $dx\,dy$ | $0$ |

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds + \iint_{S_2} \vec{F}\cdot\hat{n}\, ds = \int_0^c\int_0^b a^2\, dy\, dz + 0$$

$$= a^2 \int_0^c (y)_0^b\, dz$$

$$= a^2 \int_0^c b\, dz$$

$$= a^2 \left[ b(z)_0^c \right]$$

$= a^2bc$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds + \iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a b^2\,dx\,dz + 0.$$

$$= b^2\int_0^c (x)_0^a$$

$$= b^2\int_0^c a\,dz$$

$$= b^2\int_0^c (x)_0^a\,dz$$

$$= b^2a\int_0^c dz.$$

$$= b^2a\,(z)_0^c$$

$$= b^2ac.$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds + \iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a c^2\,dx\,dy. + 0$$

$$= c^2\int_0^b (x)_0^a\,dy.$$

$$= c^2\int_0^b a\,dy.$$

$$= c^2a\int_0^b dy$$

$$= c^2ab.$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = a^2bc + b^2ac + c^2ab.$$

$$= abc\,[a+b+c] \quad \text{--- (2)}$$

from (1) & (2) LHS = RHS

Hence verified.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