

PART A

1. If $\phi = x^2 + y^2 + z^2$, find $\nabla\phi$ at $(1,1,-1)$

Given, $\phi = x^2 + y^2 + z^2$ ----- (i)

There fore $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$ ----- (ii)

From (i), $\frac{\partial\phi}{\partial x} = 2x$; $\frac{\partial\phi}{\partial y} = 2y$; $\frac{\partial\phi}{\partial z} = 2z$ -----(iii)

Sub (iii) in (i), we get

$$\nabla\phi = \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

There fore, $(\nabla\phi)_{\text{at}(1,1,-1)} = 2\vec{i} + 2\vec{j} - 2\vec{k}$

2. Find grad r^n , where $r = |\vec{r}|$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Given, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$
 ----- (i)

Diff (i) partially w.r.t 'x'

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \text{grad} r^n = \nabla r^n$$

$$= \sum \vec{i} \frac{\partial}{\partial x} (r^n)$$

$$= \sum \vec{i} \frac{\partial}{\partial x} (r^n) \cdot \frac{\partial r}{\partial x}$$

$$= \vec{i} \cdot n \cdot r^{n-1} \cdot \frac{x}{r}$$

$$= \vec{i} \cdot n \cdot r^{n-2} \cdot x$$

$$= n \cdot r^{n-2} [x\vec{i} + y\vec{j} + z\vec{k}]$$

$$= n \cdot r^{n-2} \vec{r}$$

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

3. Find the unit vector normal to the surface $x^2 + y^2 - z = 10$ at $(1,1,1)$.

Given $\phi = x^2 + y^2 - z = 10$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\therefore (\nabla\phi)_{\text{at}(1,1,1)} = 2\vec{i} + 2\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+1} = \sqrt{9} = 3$$

Unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3}$

4. Find the directional derivative of $\phi = xy + yz + xz$ at the point $(1,2,3)$ in the direction $3\vec{i} + 4\vec{j} + 5\vec{k}$.

Given, $\phi = xy + yz + xz$ -----(i)

Let $\vec{n} = 3\vec{i} + 4\vec{j} + 5\vec{k}$ ----- (ii)

Directional derivative = $(\nabla\phi) \cdot \hat{n}$ ----- (A)

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

From (i), $\frac{\partial\phi}{\partial x} = y + z$; $\frac{\partial\phi}{\partial y} = x + z$; $\frac{\partial\phi}{\partial z} = y + x$

$$\therefore \nabla\phi = (y + z)\vec{i} + (x + z)\vec{j} + (y + x)\vec{k}$$

$$\therefore (\nabla\phi)_{\text{at}(1,2,3)} = 5\vec{i} + 4\vec{j} + 3\vec{k}$$

From (ii), we have

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\vec{i} + 4\vec{j} + 5\vec{k}}{\sqrt{50}} \text{ ----- (iv)}$$

Sub (iii) and (iv) in (A), we get

$$\begin{aligned} \text{Directional derivative} &= (\nabla\phi) \cdot \hat{n} = (5\vec{i} + 4\vec{j} + 3\vec{k}) \cdot \frac{3\vec{i} + 4\vec{j} + 5\vec{k}}{\sqrt{50}} \\ &= \frac{15 + 16 + 15}{\sqrt{25 \times 2}} = \frac{46}{5\sqrt{2}} \end{aligned}$$

5. In what direction from the point $(1,-1,-2)$ is the directional derivative

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

of $\phi = x^3y^3z^3$ a maximum? What is the magnitude of this maximum?

Given, $\phi = x^3y^3z^3$ -----(i)

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

From (i), $\frac{\partial\phi}{\partial x} = 3x^2y^3z^3$; $\frac{\partial\phi}{\partial y} = 3x^3y^2z^3$; $\frac{\partial\phi}{\partial z} = 3x^3y^3z^2$

$$\therefore \nabla\phi = 3x^2y^3z^3\vec{i} + 3x^3y^2z^3\vec{j} + 3x^3y^3z^2\vec{k}$$

$$\therefore (\nabla\phi)_{\text{at}(1,2,3)} = 24\vec{i} - 24\vec{j} - 12\vec{k}$$

There fore the directional derivative is maximum in the direction $24\vec{i} - 24\vec{j} - 12\vec{k}$.

Magnitude of this maximum is $|\nabla\phi|$

$$= \sqrt{(24)^2 + (-24)^2 + (-12)^2}$$

$$= \sqrt{1296} = 36$$

6. Find the angle between the normal to the surface $xy = z^2$ at the points (1,4,2) and (-3,-3,3).

Let $\phi = xy - z^2$ -----(i)

$$\therefore \nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$$

Normal to the surface is $\nabla_1\phi$ and $\nabla_2\phi$

$$\therefore \nabla_1\phi = (\nabla\phi)_{\text{at}(1,4,2)} = 4\vec{i} + \vec{j} - 4\vec{k}$$

$$\nabla_2\phi = (\nabla\phi)_{\text{at}(-3,-3,3)} = -3\vec{i} - 3\vec{j} - 6\vec{k}$$

$$\therefore |\nabla_1\phi| = \sqrt{33}; |\nabla_2\phi| = \sqrt{54}$$

There fore angle between the normal to the surface is,

$$\cos\theta = \frac{(\nabla_1\phi)(\nabla_2\phi)}{|\nabla_1\phi||\nabla_2\phi|} = \frac{(4\vec{i} + \vec{j} - 4\vec{k}) \cdot (-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{33}\sqrt{54}}$$

$$= \frac{9}{\sqrt{1782}} = \frac{9}{9\sqrt{22}} = \frac{1}{\sqrt{22}}$$

$$\therefore \theta = \cos^{-1}\left[\frac{1}{\sqrt{22}}\right]$$

7. If ϕ is a scalar point function, then prove that $\text{curl}(\text{grad}\phi) = 0$.

$$\text{grad}\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\text{curl grad}\phi = \nabla \times \left[\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \right]$$

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

$$\begin{aligned}
 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial x} \end{vmatrix} \\
 &= \vec{i} \left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z} \right] - \vec{j} \left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z} \right] + \vec{k} \left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} \right] \\
 &= \mathbf{0}
 \end{aligned}$$

8. If \vec{A} is a constant vector, prove that $\text{div } \vec{A} = \mathbf{0}$.

Let $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

Where A_1, A_2, A_3 are constants

$$\begin{aligned}
 \therefore \text{div } \vec{A} &= \nabla \cdot \vec{A} \\
 &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (A_1\vec{i} + A_2\vec{j} + A_3\vec{k}) \\
 &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} = \mathbf{0+0+0} \\
 \text{div } \vec{A} &= \mathbf{0}
 \end{aligned}$$

9. If \vec{A} is a constant vector, prove that $\text{curl } \vec{A} = \mathbf{0}$.

Let $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

Where A_1, A_2, A_3 are constants

$$\begin{aligned}
 \text{curl } \vec{A} &= \nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \\
 &= \vec{i} \left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right] - \vec{j} \left[\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right] + \vec{k} \left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right] \\
 &= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) \\
 \text{curl } \vec{A} &= \mathbf{0}
 \end{aligned}$$

10. Determine $f(r)$ so that the vector $f(r)\vec{r}$ is solenoidal.

Since $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\mathbf{f(r)} = xf(r)\vec{i} + yf(r)\vec{j} + zf(r)\vec{k}$$

$$\text{div } [f(r)] = \frac{\partial}{\partial x}[xf(r)] + \frac{\partial}{\partial y}[yf(r)] + \frac{\partial}{\partial z}[zf(r)]$$

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

$$\begin{aligned}
 &= f(r) + xf'(r) \frac{\partial r}{\partial x} + yf'(r) \frac{\partial r}{\partial y} + f(r) + f(r) + zf'(r) \frac{\partial r}{\partial z} \\
 &= 3f(r) + f'(r) \left[x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z} \right] \\
 &= 3f(r) + f'(r) \left[x \frac{x}{r} + y \frac{y}{r} + z \frac{z}{r} \right] \\
 &= 3f(r) + \frac{f'(r)}{r} [x^2 + y^2 + z^2] \\
 &= 3f(r) + rf'(r)
 \end{aligned}$$

Since $f(r)\vec{r}$ is solenoidal, $\text{div}[f(r)\vec{r}] = 0$

ie., $3f(r) + rf'(r) = 0$

$$\frac{f'(r)}{f(r)} = \frac{-3}{r}$$

Integrating w.r.t r , we get

$$\log f(r) = -3\log r + \log c$$

$$\log f(r) = \log cr^{-3}$$

$$f(r) = cr^{-3}$$

$$f(r) = \frac{c}{r^3}$$

11. Find the value of 'a' so that the vector, $\vec{F} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + az)\vec{k}$ is Solenoidal.

Given \vec{F} is solenoidal.

$$\text{div } \vec{F} = 0$$

ie., $\nabla \cdot \vec{F} = 0$

$$\text{ie., } \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot [(x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + az)\vec{k}] = 0$$

$$\text{ie., } \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + az) = 0 \Rightarrow 1 + 1 + a = 0 \Rightarrow a = -2$$

12. Show that the vector $2xy\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 1)\vec{k}$ is irrotational.

Let $\vec{F} = 2xy\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 1)\vec{k}$

A vector \vec{F} is said to be irrotational if $\nabla \times \vec{F} = 0$

$$\text{Now, } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & (x^2 + 2yz) & (y^2 + 1) \end{vmatrix}$$

23MAT102-Complex Analysis and Laplace Transform

VECTOR CALCULUS

$$\begin{aligned}
 &= \\
 &\vec{i} \left[\frac{\partial(y^2+1)}{\partial y} - \frac{\partial(x^2+2yz)}{\partial z} \right] - \vec{j} \left[\frac{\partial(y^2+1)}{\partial x} - \frac{\partial(2xy)}{\partial z} \right] + \vec{k} \left[\frac{\partial(x^2+2yz)}{\partial x} - \frac{\partial(2xy)}{\partial y} \right] \\
 &= \vec{i}(2y-2y) - \vec{j}(0-0) + \vec{k}(2x-2x) \\
 \nabla \times \vec{F} &= \mathbf{0}
 \end{aligned}$$

13. Show that the vector $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal.

We know that, if \vec{F} is solenoidal, we have

$$\begin{aligned}
 \text{div } \vec{F} &= \nabla \cdot \vec{F} \\
 &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot [3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}] \\
 &= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2) \\
 &= \mathbf{0+0+0}
 \end{aligned}$$

$$\therefore \text{div } \vec{F} = \mathbf{0}$$

Hence \vec{F} is solenoidal.

14. Define the line integral.

Let \vec{F} be a vector field in space and let AB be a curve described in the sense A to B. Divide the curve AB into n elements $d\vec{r}_1, d\vec{r}_2, \dots, d\vec{r}_n$.

Let $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$ be the values of this vector at the junction points of the vectors $d\vec{r}_1, d\vec{r}_2, \dots, d\vec{r}_n$, then the sum

$$\lim_{n \rightarrow \infty} \sum_A^B \vec{F}_n \cdot d\vec{r}_n = \int_A^B \vec{F} \cdot d\vec{r} \quad \text{is called the line integral.}$$

If the line integral is along the curve c then it is denoted by

$$\int_c \vec{F} \cdot d\vec{r} \quad \text{or} \quad \oint_c \vec{F} \cdot d\vec{r} \quad \text{if } c \text{ is a closed curve.}$$

15. Evaluate $\int_c \vec{F} \cdot d\vec{r}$ along the curve c in xy plane, $y = x^3$ from the point (1,1) to (2,8) if $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$.

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

Given $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$, $y = x^3$

Now, $\vec{r} = x\vec{i} + y\vec{j}$; $d\vec{r} = dx\vec{i} + dy\vec{j}$

Here $y = x^3$; $dy = 3x^2 dx$

$$\begin{aligned} \therefore \int_c \vec{F} d\vec{r} &= \int_c [(5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}] \cdot [dx\vec{i} + dy\vec{j}] \\ &= \int_c [(5xy - 6x^2)dx + (2y - 4x)dy] \\ &= \int_c [(5x(x^3) - 6x^2)dx + [(2x^3 - 4x)3x^2 dx]] \\ &= \int_c (5x^4 - 6x^2 + 6x^5 - 12x^3)dx \\ &= x^5 - 2x^3 + x^6 - 3x^4 \end{aligned}$$

There fore $\int_c \vec{F} d\vec{r}$ **from the point (1,1) to (2,8)**

ie., $\int_1^2 \vec{F} d\vec{r} = [x^5 - 2x^3 + x^6 - 3x^4]_1^2 = 35$

16. Define surface integral.

An integral which is evaluated over a surface is called a surface integral.

$\therefore \lim_{n \rightarrow \infty} \sum_{i=1}^n \vec{F}(x_i, y_i, z_i) \cdot \hat{n}_i \Delta S_i$ **is known as the surface integral.**

17. Find $\iint_s \vec{r} \cdot d\vec{s}$, where s is the surface of the tetrahedron whose

vertices are (0,0,0), (1,0,0), (0,1,0), (0,0,1).

By Gauss divergence theorem,

$$\begin{aligned} \iint_s \vec{r} \cdot d\vec{s} &= \iiint_v (\nabla \cdot \vec{r}) dv \\ \nabla \cdot \vec{r} &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot [x\vec{i} + y\vec{j} + z\vec{k}] = 1+1+1 = 3 \\ \therefore \iint_s \vec{r} \cdot d\vec{s} &= \iiint_v 3dv = 3v \end{aligned}$$

18. If $\vec{F} = \text{curl} \vec{A}$, prove $\iint_s \vec{F} \cdot \hat{n} ds = 0$, for any closed surface S.

By Gauss divergence theorem,

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

$$\begin{aligned}\iint_S \vec{F} \cdot \hat{n} \, ds &= \iiint_V \nabla \cdot \vec{F} \, dV = \iiint_V \operatorname{div}(\vec{F}) \, dV \\ &= \iiint_V \operatorname{div}(\operatorname{curl} \vec{A}) \, dV = 0 \quad [\text{since } \operatorname{div}(\operatorname{curl} \vec{A}) = 0]\end{aligned}$$

19. Define Volume integral.

An integral which can be evaluated over a volume closed by a surface is called a volume integral. Volume integral can be evaluated by triple integration.

$$\text{ie., } \iiint_V f(x, y, z) \, dV$$

20. State Gauss Divergence theorem.

If \vec{F} is a vector point function, finite and differentiable in a region r bounded by a closed surface S , then the surface integral of the normal component of \vec{F} taken over S is equal to the integral of divergence of \vec{F} taken over V .

$$\text{ie., } \iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dV \quad \text{Where } \hat{n} \text{ is the unit vector in the positive normal to } S.$$

21. Evaluate $\iint_S \vec{r} \cdot \hat{n} \, ds$, where S is a Closed surface .

By Gauss Divergence theorem , we have

$$\begin{aligned}\iint_S \vec{r} \cdot \hat{n} \, ds &= \iiint_V \nabla \cdot \vec{r} \, dV \\ &= \iiint_V \left[\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right] (x \vec{i} + y \vec{j} + z \vec{k}) \, dV \\ &= \iiint_V \left[\frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} \right] \, dV \\ &= \iiint_V (1 + 1 + 1) \, dV = 3 \iiint_V dV = 3V\end{aligned}$$

22.. Prove that $\iint_S \phi \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{\phi} \, dV$

23MAT102-Complex Analysis and Laplace Transform VECTOR CALCULUS

By Gauss Divergence theorem , we have $\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} dV$

Let $F = \phi \vec{c}$ where \vec{c} is a constant vector. Then ,

$$\iint_S \vec{\phi} \vec{c} \cdot \hat{n} ds = \iiint_V \nabla \cdot (\phi \vec{c}) dv$$

$$\iint_S \vec{c} \cdot (\phi \hat{n}) ds = \iiint_V \vec{c} \cdot (\nabla \phi) dv$$

Taking \vec{c} outside the integrals , we get

$$\vec{c} \cdot \iint_S \phi \hat{n} ds = \vec{c} \cdot \iiint_V \nabla \phi dv$$

$$\iint_S \phi \hat{n} ds = \iiint_V \nabla \phi dv$$

23. Evaluate $\iint_S xdydz + ydzdx + zdx dy$ over the region of radius a.

$$\begin{aligned} \iint_S xdydz + ydzdx + zdx dy &= \iiint_V \left[\frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} \right] dx dy dz \\ &= \iiint_V (1 + 1 + 1) dx dy dz \\ &= 3 \iiint_V dv = 3v \\ &= 3 \left[\frac{4}{3} \pi a^3 \right] = 4\pi a^3 \end{aligned}$$

24. State Green's theorem in the plane

If R is a closed region of the xy-plane bounded by a simple closed curve C and if M and N are continuous derivatives in R, then

$$\int M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \text{ where C is travelled in the}$$

anti-clockwise direction.

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

25. Using Green's theorem, prove that the area enclosed by a simple closed curve C

$$\text{is } \frac{1}{2} \int (xdy - ydx) dx dy .$$

Consider By Green's theorem,

$$\int Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \dots\dots\dots(1)$$

$$\text{Consider } \frac{1}{2} \int (xdy - ydx) dx dy = \int \frac{x}{2} dy - \frac{y}{2} dx = \int -\frac{y}{2} dx + \frac{x}{2} dy$$

$$[\text{since, } M = -\frac{y}{2}; ; N = \frac{x}{2}]$$

$$\text{From (1), } \int -\frac{y}{2} dx + \frac{x}{2} dy = \iint_R \left[\frac{1}{2} - \left(-\frac{1}{2} \right) \right] dx dy$$

$$= \iint_R dx dy = \text{Area bounded by a closed curve 'C'}$$

26. State Stoke's theorem.

If \vec{F} is any continuous differentiable vector function and S is a surface enclosed by a curve C then, $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} ds$ where \hat{n} is the unit normal vector at any point of S.

27. Using Stoke's theorem, prove that $\int_c \vec{r} \cdot d\vec{r} = 0$.

$$\text{Given, } \int_c \vec{r} \cdot d\vec{r} \text{ where } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\therefore \int_c \vec{r} \cdot d\vec{r} = \iint_s \text{curl } \vec{r} \cdot \hat{n} ds \quad [\because \text{by Stoke's theorem}]$$

$$= 0 \quad \left[\because \text{curl } \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 \right]$$

28. Find the constants a,b,c so that, $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

Given $\nabla_x \vec{F} = 0$

ie.,
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\Rightarrow \vec{i} \left[\frac{\partial}{\partial y} (4x+cy+2z) - \frac{\partial}{\partial z} (bx-3y-z) \right]$$

$$- \vec{j} \left[\frac{\partial}{\partial x} (4x+cy+2z) - \frac{\partial}{\partial z} (x+2y+az) \right]$$

$$+ \vec{k} \left[\frac{\partial}{\partial x} (bx-3y-z) - \frac{\partial}{\partial y} (x+2y+az) \right] = 0$$

$$\Rightarrow \vec{i}[c+1] - \vec{j}[4-a] + \vec{k}[b-2] = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\Rightarrow \quad \mathbf{c + 1 = 0} \quad \mathbf{4 - a = 0} \quad \mathbf{b - 2 = 0}$$

$$\Rightarrow \quad \mathbf{c = -1} \quad ; \quad \mathbf{a = 4} \quad ; \quad \mathbf{b = 2}$$

29. If $\vec{F} = x^2\vec{i} + xy^2\vec{j}$, **evaluate the line integral** $\int_c \vec{F} \cdot d\vec{r}$ **from (0,0) to (1,1)**

along the path $y = x$.

Given $\vec{F} = x^2\vec{i} + xy^2\vec{j}$, $\mathbf{x = y}$

$$\mathbf{dx = dy}$$

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F} \cdot d\vec{r} = x^2 dx + xy^2 dy = x^2 dx + x^3 dx \quad [\because x = y, dx = dy]$$

$$= (x^2 + x^3) dx$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 (x^2 + x^3) dx = \frac{7}{12}$$

30. What is the greatest rate of increase of $\phi = xyz^2$ **at (1,0,3).**

Given $\phi = xyz^2$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$$

$$(\nabla\phi)_{(1,0,3)} = \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$$

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

The greatest rate of increase = $|\nabla\phi| = \sqrt{81} = 9$ units

31. Using Green's theorem, find the area of a circle of radius r .

We know by Green's theorem,

$$\text{Area} = \frac{1}{2} \int_c (x dy - y dx)$$

For a circle of radius r , we have $x^2 + y^2 = r^2$

Put $x = r \cos \theta, y = r \sin \theta$

$$dx = -r \sin \theta d\theta, dy = r \cos \theta d\theta \quad [\theta \text{ varies from } 0 \text{ to } 2\pi]$$

$$\begin{aligned} \text{Area} &= \frac{1}{2} \int_0^{2\pi} [r \cos \theta r \cos \theta - r \sin \theta (-r \sin \theta)] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} r^2 d\theta = \frac{1}{2} r^2 [\theta]_0^{2\pi} \end{aligned}$$

$$\text{Area} = \pi r^2 \text{ sq. units.}$$

32. If $\nabla\phi$ is solenoidal find $\nabla^2\phi$.

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is solenoidal.

$$\therefore \nabla \cdot \nabla\phi = 0$$

$$\text{But } \nabla^2\phi = \nabla \cdot \nabla\phi = 0$$

33. If $\vec{r} = (x\vec{i} + y\vec{j} + z\vec{k})$, find $\nabla \times \vec{r}$

$$\text{Given } \vec{r} = (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}$$

34. Define Volume integral.

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

An integral which can be evaluated over a volume closed by a surface is called a volume integral. Volume integral can be evaluated by triple integration. Ie., $\iiint_V f(x, y, z) dv$

35. State Gauss Divergence theorem.

If \vec{F} is a vector point function, finite and differentiable in a region r bounded by a closed surface S , then the surface integral of the normal component of \vec{F} taken over S is equal to the integral of divergence of \vec{F} taken over V .

ie., $\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} dv$ Where \hat{n} is the unit vector in the positive normal to S .

36. Evaluate $\iint_S \vec{r} \cdot \hat{n} ds$, where S is a Closed surface .

By Gauss Divergence theorem , we have

$$\begin{aligned} \iint_S \vec{r} \cdot \hat{n} ds &= \iiint_V \nabla \cdot \vec{r} dv \\ &= \iiint_V \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] (x i + y j + z k) dv \\ &= \iiint_V \left[\frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} \right] dv \\ &= \iiint_V (1 + 1 + 1) dv = 3 \iiint_V dv = 3V \end{aligned}$$

37. Prove that $\iint_S \phi \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{\phi} dv$ By Gauss Divergence theorem , we have

$$\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} dv$$

Let $F = \phi \vec{c}$ where \vec{c} is a constant vector. Then ,

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

$$\iint_S \vec{\phi} \cdot \vec{c} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot (\vec{\phi} \vec{c}) \, dv$$

$$\iint_S \vec{c} \cdot (\vec{\phi} \hat{n}) \, ds = \iiint_V \vec{c} \cdot (\nabla \phi) \, dv$$

Taking \vec{c} outside the integrals , we get

$$\vec{c} \cdot \iint_S \vec{\phi} \cdot \hat{n} \, ds = \vec{c} \cdot \iiint_V \nabla \phi \, dv$$

$$\iint_S \vec{\phi} \cdot \hat{n} \, ds = \iiint_V \nabla \phi \, dv$$

38. Evaluate $\iint_S xdydz + ydzdx + zdx dy$ over the region of radius a.

$$\iint_S xdydz + ydzdx + zdx dy = \iiint_V \left[\frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} \right] dx dy dz$$

$$= \iiint_V (1 + 1 + 1) dx dy dz$$

$$= 3 \iiint_V dv = 3v$$

$$= 3 \left[\frac{4}{3} \pi a^3 \right] = 4\pi a^3$$

39. State Green's theorem in the plane

If R is a closed region of the xy-plane bounded by a simple closed curve C and if M and N are continuous derivatives in R, then

$$\int M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \text{ where C is travelled in the anti-}$$

clockwise direction.

40. Using Green's theorem , prove that the area enclosed by a simple closed curve C

23MAT102-Complex Analysis and Laplace Transform
VECTOR CALCULUS

is $\frac{1}{2} \int (xdy - ydx) dx dy$.

consider By Green's theorem,

$$\int Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \dots\dots\dots(1)$$

Consider $\frac{1}{2} \int (xdy - ydx) dx dy = \int \frac{x}{2} dy - \frac{y}{2} dx = \int -\frac{y}{2} dx + \frac{x}{2} dy$

[since, $M = -\frac{y}{2}$; $N = \frac{x}{2}$]

From (1), $\int -\frac{y}{2} dx + \frac{x}{2} dy = \iint_R \left[\frac{1}{2} - \left(-\frac{1}{2} \right) \right] dx dy$

$$= \iint_R dx dy = \text{Area bounded by a closed curve 'C'}$$

41. State Stoke's theorem.

If \vec{F} is any continuous differentiable vector function and S is a surface enclosed by a curve C then, $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} ds$ where \hat{n} is the unit normal vector at any point of S.

42. If $\vec{F} = (y^2 \cos x + z^2) \vec{i} + (2y \sin x - 4) \vec{j} + 3xz^2 \vec{k}$, find its scalar potential.

To find ϕ such that $\vec{F} = \text{grad } \phi$

$$(y^2 \cos x + z^2) \vec{i} + (2y \sin x - 4) \vec{j} + 3xz^2 \vec{k} = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

Integrating the equations partially w.r.to x,y,z respectively.

$$\phi = y^2 \sin x + xz^3 + f_1(y, z)$$

$$\phi = y^2 \sin x - 4y + f_2(x, z)$$

$$\phi = xz^3 + f_3(y, z)$$

Therefore $\phi = y^2 \sin x + xz^3 - 4y + c$ is a scalar potential.