SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

19MAT204 – PROBABILITY AND STATISTICS

PART-A (TWO MARK QUESTIONS)

1. The bivariate random variable X and Y has the pdf

f(x,y)={ $f(x,y)=\begin{cases} kx^2(8-y), x<y<2x \\ 0\le x<2 \end{cases}$ find k.

Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dydx=1$$

$$\int_{0}^{2}\int_{x}^{2x} kx^2(8-y)\,dydx=1 \qquad k\int_{0}^{2} x^2\left(8y-\frac{y^2}{2}\right)_{x}^{2x}dx=1$$

$$k\int_{0}^{2} x^2\left(16x-\frac{4x^2}{2}-8x+\frac{x^2}{2}\right)dx=1 \quad k\int_{0}^{2}\left(16x^3-2x^4-8x^3+\frac{x^4}{2}\right)dx=1$$

$$k\int_{0}^{2}\left(8x^3-\frac{3x^4}{2}\right)dx=1 \quad k\left[\frac{8x^4}{4}-\frac{3x^5}{10}\right]_0^2=1$$

$$k\left[32-\frac{48}{5}\right]=1 \qquad k\left[\frac{112}{5}\right]=1$$

$$k\left[\frac{112}{5}\right]=1$$

$$k=\frac{5}{112}$$

2. The joint pdf of random variable x and y is given by $f(x,y)=kxye^{-(x^2+y^2)}, x>0, y>0$ find the value of k.

Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} kxye^{-(x^2+y^2)}\,dy\,dx = 1$$

$$k\int_{0}^{\infty} ye^{-y^2}\,dy \int_{0}^{\infty} xe^{-x^2}\,dx = 1 \qquad \left[\int_{0}^{\infty} xe^{-x^2}\,dx = \frac{1}{2}\right]$$

$$k\frac{1}{2}.\frac{1}{2} = 1, \quad k = 4$$

3. If X and Y have joint pdf $f(x,y) = \begin{cases} x+y, 0<x<1, 0<y<1 \\ 0, \quad \text{otherwise} \end{cases}$. check whether X and Y are independent.

Ans:

The marginal density of X is

$$f(x)=\int_{-\infty}^{\infty} f(x,y)dy \qquad f(x)=\int_{0}^{1}(x+y)dy$$

$$f(x)=\left[xy+\frac{y^2}{2}\right]_0^1 \qquad f(x)=x+\frac{1}{2}$$

$$f(y)=\int_{-\infty}^{\infty} f(x,y)dy \qquad f(y)=\int_{0}^{1}(x+y)dy$$

$$f(y)=\left[\frac{x^2}{2}+xy\right]_0^1 \qquad f(y)=\frac{1}{2}+y$$

$$f(x).f(y)=\left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \qquad f(x).f(y)\neq f(x,y)$$

4. Let X and Y have j.d.f f(x,y)=2, 0<x<y<1. Find m.d.f

   Ans:

   Marginal density of X is given by

$$f(x)=\int_{-\infty}^{\infty} f(x,y)dy \qquad =\int_{x}^{1} 2dy$$

$$=2[y]_x^1$$

$$=2(1-x),0<x<1.$$

   Marginal density function of Y is given by

$$f(y)=\int_{-\infty}^{\infty} f(x,y)dy \qquad =\int_{0}^{y} 2dx \qquad =2[x]_0^y \qquad =2y,0<y<1.$$

5. The j.d.f of the random variables X and Y is given by

$$f(x,y)=\begin{cases} 8xy, 0<x<1, 0<y<x \\ 0, \quad \text{otherwise} \end{cases} \text{. find } f_x(x).$$

Ans:

$$f_x(x)=\int_{-\infty}^{\infty} f(x,y)dy$$

$$=\int_0^x 8xydy \qquad =8x\left(\frac{y^2}{2}\right)_0^x$$

$$=8x\left(\frac{x^2}{2}\right)$$

$$f_x(x)=4x^3, 0<x<1$$

6. Given $f(x,y)=\begin{cases} cx(x-y), 0<x<2, -x<y<x \\ 0, \quad \text{otherwise} \end{cases}$, find c.

Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx=1$$

$$\int_0^2\int_{-x}^{x} cx(x-y)dydx=1$$

$$c\int_0^2 \left[x^2y - x.\frac{y^2}{2}\right]_{-x}^{x} dx = 1$$

$$c\int_0^2 \left[x^3 - \frac{x^2}{2} + x^3 + \frac{x^3}{2}\right] dx = 1$$

$$c\int_0^2 2x^3 dx = 1$$

$$2c\left[\frac{x^4}{4}\right]_0^2 = 1 \qquad 2c\left[\frac{16}{4}\right] = 1 \qquad c = \frac{1}{8}$$

6. The joint p.d.f of a bivariate random variable (X,Y) is given by

$$f(x,y) = \begin{cases} kxy, 0 < x < 1, 0 < y < 1 \\ 0, \quad otherwise \end{cases}$$, find K.

Ans:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$

$$\int_0^1\int_0^1 kxydxdy = 1 \qquad k\int_0^1 \left[\frac{x^2}{2}y\right]_0^1 dy = 1$$

$$k\int_0^1 \frac{y}{2}dy = 1 \qquad k\left[\frac{y^2}{4}\right]_0^1 = 1$$

$$k = 4$$

7. If the joint pdf of (x,y) is $f(x,y)=\frac{1}{4}, 0 < x, y < 1$, find $p(x+y \le 1)$.

Ans:

$$p(x+y \le 1) = p(x \le 1-y)$$

$$= \int_0^1 \int_0^{1-y} f(x,y)\,dxdy$$

$$= \int_0^1 \int_0^{1-y} \frac{1}{4}\,dxdy \qquad = \frac{1}{4}\int_0^1 [x]_0^{1-y}\,dy$$

$$= \frac{1}{4}\int_0^1 [(1-y)]\,dy \qquad = \frac{1}{4}\left[y - \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{4}\left[1 - \frac{1}{2}\right] \qquad = \frac{1}{4}\cdot\frac{1}{2} \qquad = \frac{1}{8}$$

8. Two random variables X and Y have joint pdf $f(x,y) = \begin{cases} \frac{xy}{96}, & 0 < x < 4, 1 < y < 5 \\ 0, & \text{otherwise} \end{cases}$, find E(x).

Ans:

$$E(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dxdy$$

$$=\int_1^5\int_0^4 x\left(\frac{xy}{96}\right)dxdy \qquad =\frac{1}{96}\int_1^5\left[y.\frac{x^3}{3}\right]_0^4 dy$$

$$=\frac{1}{96}\int_1^5\frac{64}{3}ydy \qquad =\frac{64}{288}\left[\frac{y^2}{2}\right]_1^5$$

$$=\frac{2}{9}\left[\frac{25}{2}-\frac{1}{2}\right] \qquad =\frac{1}{9}(24)$$

$$=\frac{8}{3}.$$

9. Let X be a Random variable with pdf $f(x)=\frac{1}{2}, -1\le x\le 1$, and let $Y=X^2$, find E(Y).

Ans:

$$Y=x^2$$
$$E(Y)=E(x^2)$$
$$\int_{-\infty}^{\infty}x^2f(x)dx=\int_{-\infty}^{\infty}x^2\left(\frac{1}{2}\right)dx=\frac{1}{2}\left(\frac{x^3}{3}\right)_{-1}^{1}=\frac{1}{6}(2)=\frac{1}{3}.$$

10. If the joint pdf of (x,y) is given by $f(x,y)=x+y, 0\le x,y\le 1$. Find $E(XY)$.

Ans:

$$E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf(x,y)dxdy$$
$$=\int_0^1\int_0^1 xy(x+y)dxdy$$

$$=\int_0^1\int_0^1\left(x^2y+xy^2\right)dxdy$$

$$=\int_0^1\left(\frac{x^3}{3}y+\frac{x^2}{2}y^2\right)_0^1 dy$$

$$=\int_0^1\left(\frac{y}{3}+\frac{y^2}{2}\right)_0^1 dy$$

$$=\left(\frac{y^2}{6}+\frac{y^3}{6}\right)_0^1=\frac{2}{6}=\frac{1}{3}.$$

11. Find the acute angle between the two lines of regression

    Ans:

    The acute angle between the two lines of regression is $\tan\theta=\frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x{}^2+\sigma_y{}^2}\right)$.

12. State the equation of the two regression lines. What is the formula for correlation coefficient

    Ans:

    X on Y is $\left(\mathbf{x}-\overline{\mathbf{x}}\right)=\mathbf{b}_{\mathbf{xy}}\left(\mathbf{y}-\overline{\mathbf{y}}\right)$ and Y on X is $\left(\mathbf{y}-\overline{\mathbf{y}}\right)=\mathbf{b}_{\mathbf{yx}}\left(\mathbf{x}-\overline{\mathbf{x}}\right)$.

    Correlation coefficient $\mathbf{r}=\sqrt{\mathbf{b}_{\mathbf{xy}}.\mathbf{b}_{\mathbf{yx}}}$.

13. If X and Y are independent random variables with variance 2 and 3. Find the variance of 3X+4Y.

    Ans:

$$Var(x)=2,\ Var(y)=3$$

$$Var(3X+4Y)=3^2Var(X)+4^2Var(Y)$$

$$=9Var(X)+16Var(Y)$$

$$=9*2+16*3$$

$$=66$$

14. The joint pdf of (X,Y) is given by $e^{-(x+y)}, 0 < x, y < \infty$. Are X and Y independent?

Ans :

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\,dy$$

$$= \int_{0}^{\infty} e^{-x} e^{-y}\,dy$$

$$= e^{-x}\left(-e^{-y}\right)_0^{\infty} = -e^{-x}(0-1) = e^{-x}.$$

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\,dx$$

$$= \int_{0}^{\infty} e^{-x} e^{-y}\,dx$$

$$= e^{-y}\left(-e^{-x}\right)_0^{\infty} = -e^{-y}(0-1) = e^{-y}.$$

$f(x)f(y) = e^{-x}e^{-y} = e^{-(x+y)} = f(x, y)$ Therefore, X any Y are independent.

15. The two lines of regression are $8x - 10y + 66 = 0$, $40x - 18y - 214 = 0$. Find the mean value of X and Y.

Ans:

$8x - 10y = -66$ ________ (1)

$40x - 18y = 214$ ________(2)

Solving (1) and (2) , we get $x = 104, y = 17$

Mean of X = 13

Mean of Y = 17.

16. The two regression lines are $x = \frac{9}{20}y + \frac{107}{20}$, $y = \frac{4}{5}x + \frac{33}{5}$. Find correlation coefficient?

Ans:

$$r = \sqrt{b_{xy}.b_{yx}}$$

Here, $b_{xy} = \frac{9}{20}$, $b_{yx} = \frac{4}{5}$

$$r = \sqrt{\frac{9}{20} X \frac{4}{5}} = 0.6$$

17. If the pdf of X is $f(x) = 2x$, $0 < x < 1$. Find the pdf of $y = 3x+1$.

Ans:

Given $y = 3x+1$

$$\frac{dy}{dx} = 3 \Rightarrow \frac{dx}{dy} = \frac{1}{3}$$

$$f_y(y) = f(x)\left|\frac{dx}{dy}\right| \qquad Y = 3X+1,$$

$$= 2x\left(\frac{1}{3}\right) \qquad 3X = Y-1$$

$$= 2\left(\frac{y-1}{3}\right)\left(\frac{1}{3}\right) \qquad X = \frac{Y-1}{3}$$

$$= \frac{2}{9}(y-1).$$

When x=0, y=1

When x=1, y=4 $\qquad f(y) = \frac{2}{9}(y-1), 1 < y < 4$

18. Write the formula for transformation of two dimensional random variable

Ans:

$$f(u,v) = |J|\, f(x,y)$$

Here, $|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

19. If $x = uv$ and $v = y$ then find $\frac{\partial(x,y)}{\partial(u,v)}$.

Ans:

Solution: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$.

20. State Central Limit theorem.

Liapounoff's form:

If $X_i\,(1,2,....,n)$ be n independent random variable such that $E(X) = \mu$ and $Var(X) = \sigma^2$, then under certain general conditions, the random variable $S_n = X_1 + X_2 + ... + X_n$ is asymptotically normal with mean μ and standard deviation σ.

Lindberg-Levy's form:

If $X_1 + X_2 + ... + X_n$ be a sequence of independent identically distributed random variables with $E(X) = \mu$, $Var(X) = \sigma^2$ and if $S_n = X_1 + X_2 + ... + X_n$, then under certain general conditions S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to infinity.