SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

19MAT204 – PROBABILITY AND STATISTICS

PART-A (TWO MARK QUESTIONS)

1. Determine the binomial distribution whose mean is 9 and whose standard deviation is $\frac{3}{2}$.

   Answer:

   np = 9 and npq = $\frac{9}{4}$. q = $\frac{npq}{np} = \frac{1}{4} \Rightarrow p = 1 - q = \frac{3}{4}$

   np = 9 ⇒ n= $9 \times \frac{4}{3} = 12$

   ∴P[X=r] = $12\, C_r \,. \left[\frac{3}{4}\right]^r \left[\frac{1}{4}\right]^{12-r}, r = 0,1,2, \ldots..12$

2. A die is thrown 3 times. If getting a '6' is considered a success, find the probability of atleast two successes.

   Answer:

   P=1/6; q= 5/6; n=3.

   P[atleast two successes] = P(2) + P(3)

   $$= 3C_2 \,. \left[\frac{1}{6}\right]^2 \frac{5}{6} + 3C_3 \,. \left[\frac{1}{6}\right]^3 = \frac{2}{27}$$

3. Find the MGF of binomial distribution.

   Answer:

   $M_x(t) = \sum_{r=0}^{n} nC_r \,.\, (pe^t)^r \,.\, q^{n-r}$

   $= (q + pe^t)^n$

4. For a random variable X, $M_x(t) = \frac{1}{81}(e^t + 2)^4$, find P[X≤ 2].

   Answer:

   $$M_x(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^4.$$

   For Binomial distribution, $M_x(t) = (q + pe^t)$

   ∴ n=4, q=2/3, p=1/3

   ∴ P[X≤2] = P(0) + P(1) + P(2)

   $$= \left(\frac{2}{3}\right)^4 + 4C_1\frac{1}{3}\left(\frac{2}{3}\right)^3 + 4C_1\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$$

   $$= \frac{1}{81}[16 + 32 + 24] = \frac{72}{81}$$

   = 0.8889

5. The mean and variance of a binomial variance are 4 and 4/3 respectively, find P [X≥ 1] .

   Answer:

   np = 4, npq = $\frac{4}{3} \Rightarrow q = \frac{1}{3}$and $p = \frac{2}{3}$ $\therefore n = 4 \times \frac{3}{2} = 6.$

   P[X≥ 1] = 1- P[X< 1] = 1 -P[X= 0]

   = 1- $\left(\frac{1}{3}\right)^6$ = 0.9986

6. If 6 of 18 new buildings in a city violate the building code, what is the probability that a building inspector, who randomly selects 4 of the new buildings for inspection, will catch none of the buildings that violate the buildings code?

Answer:

P= probability that a building violates building code.

⇨ P= $\frac{6}{18} = \frac{1}{3} \therefore q = \frac{2}{3}$ here n = 4,

Required probability = $q^4 = \left(\frac{2}{3}\right)^4$ = 0.1975

7. For a binomial distribution, mean is 6 and standard deviation is $\sqrt{2}$. Find the first two terms of the distribution.

Answer:

np = 6, npq = 2; $q = \frac{2}{3} \Rightarrow q = \frac{1}{3} \therefore p = \frac{2}{3}$. Here n = 9.

The first two terms are $\left(\frac{1}{3}\right)^9, 9C_1\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8$

8. A certain rare blood can be found in only 0.05% of people. If the population of a randomly selected group is 3000, what is the probability that atleast 2 people in the group have this rare blood type ?

Answer:

P=0.05% => p=0.0005; n = 3000; $\lambda = np$

⇒ λ=3000 x $\frac{5}{10000}$= 1.5

$P[X \geq 2]$ = 1- P(X<2) = 1- P(X=1)

= 1- $e^{-1.5}\left(1 + \frac{1.5}{1!}\right) = 0.4422$

9. It is known that 5% of the books bound at a certain bindery have defective bindings. Find the probability that 2 of 100 books bound by this bindery will have defective bindings.

Answer:

$\lambda = np \Rightarrow \lambda = 100 \times 5/100 = 5$

$\therefore P[X=2] = \frac{5^2 e^{-5}}{2!} = 0.084$

10. If X is a poisson variate such that P(X=2) = 9P(X=4) + 90P(X=6), find the variance .

Answer:

$\frac{e^{-\lambda}\lambda^2}{2!} = \frac{9e^{-\lambda}\lambda^4}{4!} + \frac{90e^{-\lambda}\lambda^6}{6!} \Longrightarrow \lambda^4 + 3\lambda^2 - 4 = 0$

$\Longrightarrow (\lambda^2 + 4)(\lambda^2 - 1) = 0$

$\therefore \lambda^2 = 1 \Longrightarrow \text{variance} = \lambda = 1.$

11. The moment generating function of a random variable X is given by $M_x(t) = e^{3(e^t - 1)}$. Find P(X=1)

Answer:

$M_x(t) = e^{\lambda(e^t-1)} = e^{3(e^t-1)} \Longrightarrow \lambda = 3$

$P(X = 1) = \lambda e^{-\lambda} \Longrightarrow P(X=1) = 3e^{-3}.$

12. State the conditions under which the position distribution is a limiting case of the Binomial distribution.

Answer:

i) $n \rightarrow \infty$

ii) $p \to 0$

iii) np= ⅄, a constant

13. Show that the sum of 2 independent poisson variates is a poisson variates.

Answer:

Let X~P($⅄_1$) and Y ~P($⅄_2$)

Then $M_x(t) = e^{⅄_1(e^t-1)}$; $M_y(t) = e^{⅄_2(e^t-1)}$

$M_{x+y}(t) = M_x(t)M_y(t) = e^{(e^t-1)(⅄_1+⅄_2)}$

$\Longrightarrow$ X + Y is also a poisson variate

14. In a book of 520 pages, 390 typo-graphical errors occur. Assuming poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.

Answer:

$$⅄ = \frac{390}{520} = 0.75$$

P (X=x) = $\frac{e^{-⅄}⅄^x}{x!} = \frac{e^{-0.75}(0.75)^x}{x!}$, x = 0.1.2,…

Required probability = [P(X = 0)]^5 = $(e^{-0.75})^5 = e^{-3.75}$

15.If X is a poisson variate such that P(X=2)= 2/3 P(X=1) evaluate P(X=3).

Answer:

$$\frac{e^{-⅄}⅄^2}{2!} = \frac{2}{3}\frac{e^{-⅄}⅄}{1!} \Longrightarrow ⅄ = \frac{4}{3}$$

$$\therefore P[X=3] = \frac{e^{-⅄}\left(\frac{4}{3}\right)^3}{3!}$$

16.If for a poisson variate X, E(X^2) $= 6$, What is E(X)?

Answer:

$\lambda^2+\lambda \;=\; 6 \Longrightarrow \lambda^2+\lambda\;$ - 6

$= 0 \Longrightarrow (\lambda+3)(\lambda-2) = 0 \Longrightarrow \lambda = 2$, -3

But $\lambda>0$, $\lambda=2$. E(X) = $\lambda= 2$

17. If X is a poisson variate with mean λ , show that E(X^2) $= \lambda\, E(X+1)$.

Answer:

E(X^2) $=\lambda^2+\lambda$

E(X+1) = E(X)+1 = λ + 1

$\therefore$ E(X^2) $= \lambda\,(\lambda+1) = \lambda\, E(X+1)$

18. What are mean and variance of the geometric distribution defined P[X=x] = q^2p, x=0,1,2,…..

Answer:

Mean = $\frac{q}{p}$ and variance = $\frac{q}{p^2}$

19.A Couple decides to make have children until they have male child. If the probability of a male child in their community is 1/3 , how many children are they expected to have before the first male child is born ?

Answer:

The waiting time for a male child has a geometric distribution with P=1/3.

Q= 1- p=2/3, Hence the expected number of children (ie., mean) = q/p = 2

20. Identify the distribution with the m.g.f. $M_x(t) = e^t(5-4e^t)^{-1}$.

Answer:

$M_x(t) = \frac{\frac{1}{5}e^t}{1-\frac{4}{5}e^t}$. If P[X=r] = p$q^{r-1}, r = 1,2, \ldots \ldots$ then

$$M_x(t)\sum_{r=1}^{\infty} e^{tr}q^{r-1}\,p = pe^t\sum_{r=1}^{\infty}(qe^t)^{r-1} \Rightarrow M_x(t) = \frac{pe^t}{1-qe^t}$$

The given MGF is the m.g.f of geometric distribution with parameter p= 1/5 whose p.m.f. is P[X=r] = $pq^{r-1}, r = 1,2, \ldots.$

21. Find the MGF of a RV which is uniformly distributed over (-1,2).

Answer:

$M_x(t) = \frac{1}{3}\int_{-1}^{2} e^{tx}\,dx = \frac{e^{2t}-e^{-t}}{3t}$ for t≠0 and $M_x(t) = \frac{1}{3}\int_{-1}^{2} dx$ =1 for t = 0

22.If X has uniform distribution in (-3,3,find P[(X-2)<2]

Answer:

P.d.f f(x)= 1/6 , -3<X<3, and =0; otherwise

P[(X-2)<2] = P[0<X<4] = $\frac{1}{6}\int_{0}^{3} dx$ = 3/6 = 1/2 .

23. If x has uniform distribution in (-a,a), a>0, find ‘a’ such that P(X<1) = P(X>1).

Answer:

P.d.f f(x)= $\frac{1}{2a}$, -a < X <a and =0 , otherwise

P[X<1] = 1/2 $\Rightarrow \int_{-1}^{1}\frac{1}{2a}dx\ \frac{1}{2} \Rightarrow \frac{1}{a} = \frac{1}{2}$ $\quad\therefore\ a = 2$

24.If the MGF of a continuous R.V X is $\frac{e^{5t}-e^{4t}}{t}$, t ≠ 0, what is the distribution of X? what are its mean and variance?

Answer:

$M_x(t)$ of uniform distribution in (a,b) is

$M_x(t) = \frac{e^{bt}-e^{at}}{(b-a)^t}$. The distribution of X is uniform in (4,5)

$Mean = \frac{b+a}{2}$ = 9/2 and variance = $\frac{(b+a)^2}{12} = \frac{1}{12}$

25. If X has geometric distribution with p.m.f P[X=r] = pq^{r-1} , r= 1,2,3,….. find P[X is odd].

Answer:

P[X is odd] = p + pq^2+ pq^4 $+ \cdots = \frac{p}{1-q^2}$ = $\frac{1}{1-q}$

26.Find the mean and the variance of the distribution P[X=x] = $2^{-x}, x = 1,2, \ldots$

Answer:

P[X=x] = $\frac{1}{2^x} = \left(\frac{1}{2}\right)^{x-1} \frac{1}{2}$, $x = 1,2, \ldots ..$

P=1/2 and q= 1/2

Mean = q/p = 1;   variance = q/p^2 = 2

27.If X is uniformly distributed with mean 1and variance 4/ 3 , find P(X<0).

Answer:

Let X~ $U(a,b)$ then $\frac{b+a}{2} = 1$ and $\frac{(b-a)^2}{12}$ = 4/3

a+b=2a nd b-a=4. Solving a = -1, b=3.

P(x) = ¼, -1<x<3. P (X<0) = $\int_{-1}^{0} p(x)dx = \frac{1}{4}$

28. The time (in hours) required to repair a machine is exponentially distributed with parameter ⋏= $\frac{1}{2}$. What is the probability that a repair takes atleast 10 hours given that its duration exceeds 9 hours ?

Answer:

Let X be the R.V which represents the time to repair the machine.

P[X$\geq 10/x \geq 9$] = P(X$\geq$ 1) (by memory less property)

= $\int_{1}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx$ = 0.6065

29. The time (in hours) required to repair a machine is exponentially distributed with parameter ⋏= $\frac{1}{3}$. What is the probability that the repair time exceeds 3 hours ?

Answer:

X- represent the time to repair the machine

P.d.f of X, f(x) = $\frac{1}{3} e^{-\frac{x}{3}}$, x>0

P(x>3) = $\int_{3}^{\infty} \frac{1}{3} e^{-\frac{x}{3}} dx = e^{-1}$ = 0.3679

30. Find the MGF of an exponential distribution with parameter ⋏.

Answer:

$M_x(t) = \lambda \int_{0}^{\infty} e^{tx} e^{-\lambda x} dx = \lambda \int_{0}^{\infty} e^{-(\lambda - x)x} dx$

$= \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1}$

31. Write down the MGF of gamma distribution and hence find its mean and variance.

Answer:

$M_x(t) = (1-t)^{-⅄} = 1 + ⅄\, t + \frac{⅄^2 + ⅄}{2!} t^2 + \cdots$

$Mean \ = ⅄;$ $E(X^2) = ⅄^2 + ⅄ \Rightarrow var\ (X) = ⅄.$

32. Mention any four properties of normal distribution ?

Answer:

(1) The curve is bell shaped

(2) Mean, Median, Mode coincide.

(3) All odd central moments vanish

(4) X-axis is an asymptote to the normal curve

33. If X is normal variate with mean 30 and S.D 5, find P[26 < X < 40]

Answer:

P [26 < X < 40] = P [-0.8 ≤ Z ≤ 2] where Z = $\frac{X-30}{5}$

= P [0 ≤ Z ≤ 0.8] + P[0 ≤ Z ≤ 2]

= 0.2881 + 0.4772 = 0.7653

34. If X is a normal variate with mean 30 and s.d5 , find P [$|X - 30| \leq 5$].

Answer:

P [$|X-30|$≤ 5] = P [25 ≤ X ≤ 35] = P [-1 ≤ Z ≤ 1]

= 2P (2 ≤ Z ≤ 1) = 2(0.3413) = 0.6826

35.X is normally distributed R.V with mean 12 and SD 4. Find P [X ≤ 20].

Answer:

P [X ≤ 20] = P [Z ≤ 2] where Z = $\frac{X-12}{4}$

= P [-∞ ≤ Z 0] + P [0 ≤ Z ≤ 2]

= 0.5 + 0.4772 = 0.9772

36.For a certain normal distribution, the first moment about 10 is 40 and the fourth moment about 50is 48. Find the mean and s.d of the distribution.

Answer:

Mean A + μ_1' ⇒ Mean = 10 + 40 = 50

μ_1'($about\ the\ point\ X = 50$) = 48 ⇒ μ_4= 48

Since mean is 50, $3\sigma^4$ = 48

σ = 2.

37.If X is normally distributed with mean 8 and s.d 4 , find P (10 ≤ X ≤ 15).

Answer:

$P(10 \le X \le 15) = P[0.5 \le X \le 1.75]$

$= P[0.5 \le X \le 1.75] - P[0 \le X \le 0.5]$

$= 0.2684$

38.X is a normal variate with mean 1 and variance 4 , Y is another normal variate independent of X with mean 2 and variance 3 , what is the distribution of X + 2Y ?

Answer:

$E[X + 2Y] = E(X) + 2E(Y) = 1 + 4 = 5$

$V[X+2Y] = V(X) + 4V(Y) = 4+4(3) = 16$

$X + 2Y \sim N(5,16)$ by additive property.

39.If X is a C.R.V with p.d.ff(x) = $\frac{x}{12}$ in 1 < x 5 and = 0 else where , find the p.d.f of Y = 2X – 3

Answer:

$Y = 2X - 3 \Rightarrow \frac{dx}{dy} = \frac{1}{2}$

$f_y(y) = f_x(x)\left|\frac{dx}{dy}\right| \Rightarrow f_y(y) = \frac{Y+3}{48}$ in $-1 < y < 7$.

40.If the continuous R.V X has p.d.f f(x) = $\frac{2(x+1)}{9}$, in -1 < X < 2 and = 0 else where , find the p.d.f of Y = X^2

Answer:

$f_y(\mathrm{y}) = \frac{2}{9\sqrt{y}}$,0<y<1

And $f_y(\mathrm{y}) = \frac{1}{9}(1+\frac{1}{\sqrt{y}})$, 1 < y < 4

41.The p.d.f of a R.V X is given by f(x) = $\begin{bmatrix} 2x, 0<x<1 \\ 0, else\ where, \end{bmatrix}$ find the p.d.f of Y = $8X^3$

Answer:

Y = $8X^3$ is strictly increasing function in (0,1)

$f_y(\mathrm{y}) = f_x(\mathrm{x})\left|\frac{dx}{dy}\right|$ where x = $\frac{1}{2}y^{1/3}$

$\Rightarrow f_y(\mathrm{y}) = \frac{1}{6}y^{-1/3}$, 0 < y < 8.

42. If X is a normal R.V with mean zero and variance σ^2, Find the p.d.f of Y=e^x.

Answer:

$f_y(Y) = f_x(x)\left|\frac{dx}{dy}\right| = \frac{1}{y} f_x(\log y)$

$= \frac{1}{\sigma y\sqrt{2\pi}}\exp[-(\log y - \mu)^2 / 2\sigma^2]$

43.If X has an exponential distribution with parameter 1 , find the pdf of y = $\sqrt{x}$.

Answer:

$f_y(Y) = f_x(x)\left|\frac{dx}{dy}\right| = 2ye^{-y^2}, y > 0$

44.If X is uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, find the pdf of Y= tan X.

Answer:

$f_y(Y) = \frac{1}{x}$; $x = \tan^{-1} y \Rightarrow \frac{dx}{dy} = \frac{1}{1+y^2}$

$f_y(Y) = f_x(x)\left|\frac{dx}{dy}\right| \Rightarrow f_y(Y) = \frac{1}{\pi(1+y)^2}, -\infty < y < \infty$

45. If X is uniformly distributed in (-1,1) , find the pdf of y = sin $\frac{\pi x}{2}$.

Answer:

$f_x(x) = \frac{1}{2}$, $-1 < x < 1; = 0$, otherwise.

$\frac{dy}{dx} = \cos\left(\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right) = \frac{dx}{dy}$

$= \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}$ for $-1 \leq y \leq 1$.

$f_y(Y) = \frac{1}{2}\left[\frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}\right] = f_y(Y) = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}$ for $-1 \leq y \leq 1$.

46.If the RV X is uniformly distributed over (-1,1), Find the density function of y = cos $\frac{\pi x}{2}$.

Answer:

$f_y(Y) = \frac{1}{\pi\sqrt{1-y^2}}$ $for\ 0 \leq y \leq 1$.

47. The pdf of a RV X is f(x) = 2x, 0 <x< 1, pdf of y= 3X + 1.

Answer:

$\frac{dx}{dy} = \frac{1}{3}$; $f_y(y)\left|\frac{dx}{dy}\right| f_x(x) \Rightarrow f_y(y) = \frac{2}{9}(y-1)$ in 1 < y < 4.