

6]. IF $x(t)$ is the input voltage to a circuit and $y(t)$ is the output voltage. $\{x(t)\}$ is a stationary r.p. with $\mu_x = 0$ and $R_{xx}(\tau) = e^{-2|\tau|}$. Find the mean μ_y and power spectrum $S_{yy}(\omega)$ of the output if the system transfer function is given by $H(\omega) = \frac{1}{\omega + 2i}$

Soln.

Given $\mu_x = 0$, $R_{xx}(\tau) = e^{-2|\tau|}$

WKT

$$y(t) = \int_{-\infty}^{\infty} h(u) x(t-u) du$$

$$E[y(t)] = E\left[\int_{-\infty}^{\infty} h(u) x(t-u) du\right]$$

$$= \int_{-\infty}^{\infty} b(u) \cdot E[x(t-u)] du$$

$$\mu_y = 0$$

$$\because \mu_x = 0$$

Input of PSD:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} (\cos \omega\tau - i \sin \omega\tau) d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} \cos \omega\tau d\tau - i \int_{-\infty}^{\infty} e^{-2|\tau|} \sin \omega\tau d\tau$$

$$= 2 \int_0^{\infty} e^{-2\tau} \cos \omega\tau d\tau - i(0)$$

$$= 2 \frac{2}{2^2 + \omega^2}$$

$$S_{xx}(\omega) = \frac{4}{4 + \omega^2}$$

$$\text{and } H(\omega) = \frac{1}{\omega + 2i} = \frac{1}{\omega + 2i} \cdot \frac{\omega - 2i}{\omega - 2i} = \frac{\omega - 2i}{4 + \omega^2}$$

$$|H(\omega)| = \sqrt{\left(\frac{\omega}{\omega^2 + 4}\right)^2 + \left(\frac{-2}{\omega^2 + 4}\right)^2} = \sqrt{\frac{\omega^2 + 4}{(\omega^2 + 4)^2}}$$

$$\Rightarrow |H(\omega)| = \frac{1}{\sqrt{\omega^2 + 4}}$$

$$\Rightarrow |H(\omega)|^2 = \frac{1}{\omega^2 + 4}$$

$$(1) \Rightarrow S_{yy}(\omega) = \left(\frac{4}{4+\omega^2} \right) \left(\frac{1}{\omega^2+4} \right)$$

$$= \frac{4}{(\omega^2+4)^2}$$

Q. A random process $x(t)$ is the input to a linear system whose impulse function is $h(t) = 2e^{-t}$, $t \geq 0$. The auto correlation function of the process is $R_{xx}(\tau) = e^{-2|\tau|}$. Find PSD of output process $y(t)$.

Soln.

Given $h(t) = 2e^{-t}$, $t \geq 0$

WKT

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

$$= \int_0^{\infty} 2e^{-t} e^{-i\omega t} dt$$

$$= 2 \int_0^{\infty} e^{-(1+i\omega)t} dt$$

$$= 2 \left(\frac{e^{-(1+i\omega)t}}{-(1+i\omega)} \right)_0^{\infty}$$

$$= \frac{-2}{1+i\omega} (0-1)$$

$$H(\omega) = \frac{2}{1+i\omega}$$

$$|H(\omega)|^2 = \frac{4}{1+\omega^2}$$

Input of PSD:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} (\cos \omega\tau - i \sin \omega\tau) d\tau$$

$$= 2 \int_0^{\infty} e^{-2\tau} \cos \omega\tau d\tau - i(0)$$

$$= 2 \frac{2}{4+\omega^2}$$

$$= \frac{4}{4+\omega^2}$$

$$\therefore \int_0^{\infty} e^{-ax} \cos bx dx = \frac{a}{a^2+b^2}$$

Output of PSD:

$$S_{yy}(\omega) = S_{xx}(\omega) * |H(\omega)|^2$$

$$= \frac{4}{4+\omega^2} \frac{4}{1+\omega^2}$$

$$= \frac{16}{(1+\omega^2)(4+\omega^2)}$$

Q7. A WSS noise process $N(t)$ has an auto correlation function $R_{NN}(\tau) = P e^{-3|\tau|}$, where P is a constant. Find its power spectrum.

Soln.:

$$\text{Given } R_{NN}(\tau) = P e^{-3|\tau|}$$

$$S_{NN}(\omega) = \int_{-\infty}^{\infty} P e^{-3|\tau|} e^{-i\omega\tau} d\tau$$

$$= P \int_{-\infty}^{\infty} e^{-3|\tau|} (\cos \omega\tau - i \sin \omega\tau) d\tau$$

$$= P \cdot 2 \int_0^{\infty} e^{-3\tau} \cos \omega\tau d\tau - i(0)$$

$$= 2P \frac{3}{3^2 + \omega^2}$$

$$= \frac{6P}{9 + \omega^2}$$