

Unit - 5

Linear Systems with Random Inputs

- * Linear time Invariant System
- * System transfer Function
- * Linear System with random inputs
- * Auto correlation and cross correlation functions of input and output

Unit-5
Linear time invariant system:
System:

A system is defined by a functional relationship between the input $x(t)$ and the output $y(t)$ as,

$$y(t) = F\{x(t)\}, \quad -\infty < t < \infty$$

Linear system:

A system with functional relationship $F\{x(t)\}$ is linear, if for any two inputs $x_1(t)$ and $x_2(t)$, the output of the system can be

$$\begin{aligned} \text{defined as } & F\{a_1 x_1(t) + a_2 x_2(t)\} \\ & = a_1 F\{x_1(t)\} + a_2 F\{x_2(t)\} \end{aligned}$$

Time invariant system:

$$\text{Let } y(t) = F\{x(t)\}.$$

If $y(t+h) = F\{x(t+h)\}$, then F is called a time invariant system of $x(t)$ and $y(t)$.

Causal system:

Suppose that the value of the output $y(t)$ at $t=t_0$ depends only on the past values of the input $x(t)$ at $t=t_0$.

$$\text{i.e., } y(t_0) = F\{x(t); t \leq t_0\}$$

Stable :

A system is stable if for every bounded input, the system gives bounded output.

Note :

$$* Y(t) = h(t) * x(t)$$

where $h(t)$ is weighting function (or)

Impulse response function and $x(t)$ is input.

$$* Y(t) = \int_{-\infty}^{\infty} h(u) x(t-u) du$$

$$Y(t) = \int_{-\infty}^{\infty} h(t-u) x(u) du$$

* $H(\omega) = F[h(\tau)] \rightarrow$ system transfer function

* Unit impulse response

$$\delta(t-a) = \begin{cases} \frac{1}{\epsilon}, & a - \frac{\epsilon}{2} \leq t \leq a + \frac{\epsilon}{2} \\ 0, & \text{otherwise} \end{cases}$$

where $\epsilon \rightarrow 0$.

Properties of Linear System:

Property 1:

If the input $x(t)$ and its output $y(t)$ are related by $y(t) = \int_{-\infty}^{\infty} h(u) x(t-u) du$, then the system is a linear time-invariant system.

Proof:

i). To prove $y(t)$ is linear.

$$\text{Let } x(t) = a_1 x_1(t) + a_2 x_2(t)$$

$$\text{Then } y(t) = \int_{-\infty}^{\infty} h(u) x(t-u) du$$

$$= \int_{-\infty}^{\infty} h(u) [a_1 x_1(t-u) + a_2 x_2(t-u)] du$$

$$= \int_{-\infty}^{\infty} h(u) a_1 x_1(t-u) du + \int_{-\infty}^{\infty} h(u) a_2 x_2(t-u) du$$

$$= a_1 \int_{-\infty}^{\infty} h(u) x_1(t-u) du + a_2 \int_{-\infty}^{\infty} h(u) x_2(t-u) du$$

$$y(t) = a_1 y_1(t) + a_2 y_2(t)$$

$\therefore y(t)$ is a linear

ii). To prove $y(t)$ is time invariant.

$$\text{Let } y(t) = \int_{-\infty}^{\infty} h(u) x(t-u) du$$

If $x(t)$ is replaced by $x(t+h)$, then

$$y(t) = \int_{-\infty}^{\infty} h(u) x(t+h-u) du$$

$$Y(t) = Y(t+h)$$

$\therefore Y(t)$ is time invariant.

Hence the system is linear time invariant system.

Property 2:

If $\{x(t)\}$ is a WSS process and if $Y(t) = \int_{-\infty}^{\infty} h(u) x(t-u) du$, then we've the following results.

a). $R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$

b). $R_{yy}(\tau) = R_{xx}(\tau) * h(-\tau)$

c). $S_{xy}(\omega) = S_{xx}(\omega) * H(\omega)$

d). $S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2$

Proof:

Given $Y(t) = \int_{-\infty}^{\infty} h(u) x(t-u) du$

a). $R_{xy}(\tau) = E[x(t) Y(t+\tau)]$

$$= E\left[x(t) \int_{-\infty}^{\infty} h(u) x(t+\tau-u) du\right]$$

$$= \int_{-\infty}^{\infty} h(u) E[x(t) x(t+\tau-u)] du$$

$$= \int_{-\infty}^{\infty} h(u) R_{xx}(\tau+u) du \quad \left\{ \because x(t) \text{ is WSS} \right\}$$

$$R_{xy}(\tau) = h(\tau) * R_{xx}(\tau) \quad (\text{By convolution})$$

$$b) R_{yy}(\tau) = E[y(t) y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(u) x(t-u) du y(t+\tau)\right]$$

$$= \int_{-\infty}^{\infty} h(u) E[x(t-u) y(t+\tau)] du$$

$$\text{put } \begin{cases} t-u = t_1 \\ t = t_1+u \end{cases} \quad \left| \quad \begin{cases} t+\tau = t_1+u+\tau \\ t = t_1+u \end{cases} \right.$$

$$= \int_{-\infty}^{\infty} h(u) E[x(t_1) y(t_1+u+\tau)] du$$

$$= \int_{-\infty}^{\infty} h(u) E[x(t_1) y(t_1+\tau+u)] du$$

$$= \int_{-\infty}^{\infty} h(u) R_{xy}(\tau+u) du$$

$$\text{take } u = -\alpha \\ du = -d\alpha$$

$$u = \infty \Rightarrow \alpha = -\infty$$

$$u = -\infty \Rightarrow \alpha = \infty$$

$$\therefore R_{yy}(\tau) = \int_{-\infty}^{\infty} h(-\alpha) R_{xy}(\tau-\alpha) (-d\alpha)$$

$$= \int_{\infty}^{-\infty} R_{xy}(\tau-\alpha) h(-\alpha) d\alpha$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau-\alpha) h(-\alpha) d\alpha$$

$$R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$$

$$\therefore y(t) = \int_{-\infty}^{\infty} h(u) x(t-u) du = x(t) * h(t)$$

c). WKT $R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$ by (a)

Taking Fourier Transform on both sides,

$$F[R_{xy}(\tau)] = F[R_{xx}(\tau) * h(\tau)] \\ = F[R_{xx}(\tau)] * F[h(\tau)]$$

$$S_{xy}(\omega) = S_{xx}(\omega) * H(\omega) \quad \text{where } H(\omega) = F[h(\tau)]$$

d). WKT $R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$ by (b)

Taking FT on both sides,

$$F[R_{yy}(\tau)] = F[R_{xy}(\tau) * h(-\tau)] \\ = F[R_{xy}(\tau)] * F[h(-\tau)]$$

$$S_{yy}(\omega) = S_{xy}(\omega) * H^*(\omega) \quad \begin{cases} \because H(\omega) = F[h(\tau)] \\ H^*(\omega) = F[h(-\tau)] \end{cases} \\ = S_{xx}(\omega) * H(\omega) * H^*(\omega) \quad \text{from (c)}$$

$$\Rightarrow S_{yy}(\omega) = S_{xx}(\omega) * |H(\omega)|^2$$

Hence proved.

property 3:

If the input to a time invariant, stable linear system is a WSS process, then the output will also be a WSS process.

(OR)

To show that if the input $\{x(t)\}$ is a WSS process, then the output $\{y(t)\}$ is a WSS process.

Proof:

1). WKT, the input and the output are related by,

$$y(t) = \int_{-\infty}^{\infty} h(u) x(t-u) du \rightarrow (1)$$

$$i). E[y(t)] = E\left[\int_{-\infty}^{\infty} h(u) x(t-u) du\right]$$

$$= \int_{-\infty}^{\infty} h(u) E[x(t-u)] du$$

$x(t)$ is WSS

$$\Rightarrow E[x(t)] = \text{Constant}$$

$$a, E[x(t-u)] = \text{Constant}$$

= a finite constant,

[\because Independent of t

System is stable]

= a constant

$$ii). R_{yy}(t, t+\tau) = E[y(t) y(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(u_1) x(t-u_1) du_1 \int_{-\infty}^{\infty} h(u_2) x(t+\tau-u_2) du_2\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1) h(u_2) E[x(t-u_1) x(t+\tau-u_2)] du_1 du_2$$

$\hookrightarrow (1)$

Since, $\{x(t)\}$ is a WSS process

$\Rightarrow E[x(t-u_1) x(t+\tau-u_2)]$ is a function of τ ,
say $g(\tau)$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1) h(u_2) g(\tau) du_1 du_2$$

$$= g(\tau) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u_1) h(u_2) du_1 du_2$$

= a function of τ

$$\text{i.e., } R_{yy}(t, t+\tau) = R_{yy}(\tau)$$

$\therefore \{y(t)\}$ is a WSS process.
