

Wiener-Khinchine Theorem:

If $x_T(\omega)$ is the FT of the truncated random process is defined as,

$$x_T(t) = \begin{cases} x(t) & \text{for } |t| \leq T \\ 0 & \text{for } |t| > T \end{cases}$$

where $[x(t)]$ is a real WSS process with power spectral density function $S_{xx}(\omega)$, then

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \left[\frac{1}{2T} E \left\{ |x_T(\omega)|^2 \right\} \right]$$

Proof:

Given $x_T(\omega)$ is the Fourier transform of $x_T(t)$

$$\begin{aligned} \Rightarrow x_T(\omega) &= \int_{-\infty}^{\infty} x_T(t) e^{-i\omega t} dt \\ &= \int_{-T}^T x(t) e^{-i\omega t} dt \end{aligned}$$

$$\begin{aligned} \text{Now, } |x_T(\omega)|^2 &= [x_T(\omega)] [x_T(\omega)]^* \quad \because |z|^2 = z \bar{z} \\ &= x_T(\omega) x_T(-\omega) \end{aligned}$$

$$= \int_{-T}^T x(t_1) e^{-i\omega t_1} dt_1 \cdot \int_{-T}^T x(t_2) e^{i\omega t_2} dt_2$$

$$= \int_{-T}^T \int_{-T}^T x(t_1) x(t_2) e^{-i\omega(t_2 - t_1)} dt_2 dt_1$$

$$[|x_T(\omega)|^2] = \int_{-T}^T \int_{-T}^T E [x(t_1) x(t_2)] e^{-i\omega(t_2 - t_1)} dt_2 dt_1$$

$$= \int_{-T}^T \int_{-T}^T R_{xx}(t_1, t_2) e^{-i\omega(t_2 - t_1)} dt_2 dt_1 \rightarrow (1)$$

$$= \int_{-T}^T \int_{-T}^T R_{xx}(t_1, -t_2) e^{-i\omega(t_1 - t_2)} dt_1 dt_2$$

$$\text{Let } t = t_2 \Rightarrow \tau = t_1 - t \Rightarrow \tau + t = t_1 \\ dt = dt_2 \quad d\tau = dt_1$$

$$E[|X_T(\omega)|^2] = \int_{-T}^T \int_{-T}^T R_{xx}(t_1 - t_2) e^{-i\omega(t_1 - t_2)} dt_1 dt_2$$

t varies from $-T$ to T

τ varies from $-T$ to T

$$\tau = -T \Rightarrow t_1 - t = -T \Rightarrow t_1 = -T + t$$

$$\tau = T \Rightarrow t_1 - t = T \Rightarrow t_1 = T + t$$

$$E[|X_T(\omega)|^2] = \int_{-T}^T \int_{-T+t}^{T+t} R_{xx}(t_1 - t_2) e^{-i\omega(t_1 - t_2)} dt dt$$

$$= \int_{-T}^T dt \int_{-T+t}^{T+t} R_{xx}(t_1 - t_2) e^{-i\omega(t_1 - t_2)} d\tau$$

$$= (t) \int_{-T}^T \int_{-T+t}^{T+t} R_{xx}(t_1 - t_2) e^{-i\omega(t_1 - t_2)} d\tau$$

$$= 2T \int_{-T+t}^{T+t} R_{xx}(t_1 - t_2) e^{-i\omega(t_1 - t_2)} d\tau$$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} E[|x_T(\omega)|^2] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T+t}^{T+t} R_{xx}(t_1 - t_2) e^{-i\omega(t_1 - t_2)} dt$$

$$= \int_{-\infty}^{\infty} R_{xx}(t_1 - t_2) e^{-i\omega(t_1 - t_2)} dt$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau$$

$$= S_{xx}(\omega)$$

Hence proved