Power Spectral Density (PSD)

The Power Spectral density of the Signal describes the power present in the Signal as a function of Frequency.

The cross power spectral density is a Spectral Analysis that compares two signals.

Definition - Power Spectral Density:

The Power Spectral Density $S_{XX}(\omega)$ of a continuous random process $X(t)$ is defined as the Fourier Transform of $R_{XX}(\tau)$

i.e.,
$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau \longrightarrow (1)$$

And $R_{XX}(\tau)$ is given by the Inverse Fourier Transform of $S_{XX}(\omega)$

i.e.,
$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\omega\tau}\, d\omega \rightarrow (2)$$

Eqns. (1) & (2) are called Weiner-Khinchine Relation.

Properties: For a WSS process,

1. The value of the spectral density function at zero frequency is equal to the total area under the graph of the autocorrelation function.
2. The Spectral density function of a real random process is an even function.

   i.e., $S_{XX}(-\omega) = S_{XX}(\omega)$
3. The mean square value of a WSS process is equal to the total area under the graph of spectral density.
4. The spectral density & the auto correlation function of a real WSS process form a Fourier Cosine transform pair.
5. A WSS random process has a non-negative power density spectrum

1). The auto correlation of a random Process is given by $R_{XX}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T}, & |\tau| \le T \\ 0, & |\tau| > T \end{cases}$

Find the power spectrum.

Soln.

Given $$R_{XX}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T}, & |\tau| \le T \\ 0, & |\tau| > T \end{cases}$$

Now, $$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) e^{-i\omega\tau}\, d\tau$$

$$= \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right)(\cos\omega\tau - i\sin\omega\tau)\, d\tau$$

$$= \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) \cos\omega\tau\, d\tau - i\int_{-T}^{T}\left(1 - \frac{|\tau|}{T}\right)\sin\omega\tau\, d\tau$$

$$= 2\int_{0}^{T} \left(1 - \frac{|\tau|}{T}\right) \cos\omega\tau\, d\tau - i(0)$$

$$= 2\left[\left(1 - \frac{\tau}{T}\right)\frac{\sin\omega\tau}{\omega} - \frac{1}{T}\frac{\cos\omega\tau}{\omega^2}\right]_0^T$$

$$= 2\left[\frac{-\cos\omega T}{T\omega^2} + \frac{1}{T\omega^2}\right]$$

$$S_{XX}(\omega) = \frac{2}{T\omega^2}(1 - \cos\omega T)$$

---

2). Find the power spectral density function whose auto correlation is given by,

$$R_{XX}(\tau) = \frac{A^2}{2}\cos\omega_0\tau$$

Soln.

Given $R_{XX}(\tau) = \frac{A^2}{2} \cos \omega_0 \tau$

Now $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$

$$= \int_{-\infty}^{\infty} \frac{A^2}{2} \cos \omega_0 \tau \, e^{-i\omega\tau} d\tau$$

$$= \frac{A^2}{2} \int_{-\infty}^{\infty} \left(\frac{e^{i\omega_0\tau} + e^{-i\omega_0\tau}}{2} \right) e^{-i\omega\tau} d\tau$$

$$= \frac{A^2}{4} \left[\int_{-\infty}^{\infty} e^{i\omega_0\tau} e^{-i\omega\tau} d\tau + \int_{-\infty}^{\infty} e^{-i\omega_0\tau} e^{-i\omega\tau} d\tau \right]$$

$$= \frac{A^2}{4} \left[\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)\tau} d\tau + \int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)\tau} d\tau \right]$$

$$= \frac{A^2}{4} \left[2\pi \delta(\omega-\omega_0) + 2\pi \delta(\omega+\omega_0) \right]$$

[By Dirac Delta Function :

$2\pi \delta(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} d\tau$]

$$= \frac{2\pi A^2}{4} \left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0) \right]$$

$$S_{XX}(\omega) = \frac{\pi A^2}{2} \left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0) \right]$$

3]. The autocorrelation function of WSS Process is given by $R_{XX}(\tau) = \alpha^2 e^{-2\alpha|\tau|}$. Determine $S_{XX}(\omega)$

Soln.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \alpha^2 e^{-2\lambda|\tau|} e^{-i\omega\tau} \, d\tau$$

$$= \alpha^2 \left[\int_{-\infty}^{0} e^{-2\lambda(-\tau)} e^{-i\omega\tau} \, d\tau + \int_{0}^{\infty} e^{-2\lambda(\tau)} e^{-i\omega\tau} \, d\tau \right]$$

$$= \alpha^2 \left\{ \int_{-\infty}^{0} e^{(2\lambda - i\omega)\tau} \, d\tau + \int_{0}^{\infty} e^{-(2\lambda + i\omega)\tau} \, d\tau \right\}$$

$$= \alpha^2 \left[\left(\frac{e^{(2\lambda - i\omega)\tau}}{2\lambda - i\omega} \right)_{-\infty}^{0} + \left(\frac{e^{-(2\lambda + i\omega)\tau}}{-(2\lambda + i\omega)} \right)_{0}^{\infty} \right]$$

$$= \alpha^2 \left[\frac{1}{2\lambda - i\omega} + \frac{1}{2\lambda + i\omega} \right]$$

$$= \alpha^2 \left[\frac{2\lambda - i\omega + 2\lambda - i\omega}{4\lambda^2 + \omega^2} \right]$$

$$S_{xx}(\omega) = \alpha^2 \left(\frac{4\lambda}{4\lambda^2 + \omega^2} \right)$$

---

4. The power spectral density of WSS process is given by, $S_{XX}(\omega) = \begin{cases} \frac{b}{a}(a - |\omega|), & |\omega| \leq a \\ 0, & |\omega| > a \end{cases}$

Soln.

Given $S_{XX}(\omega) = \begin{cases} b/a\,(a - |\omega|), & |\omega| \leq a \\ 0, & |\omega| > a \end{cases}$

Now, $$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-a}^{a} \frac{b}{a}(a - |\omega|) e^{i\omega\tau} \, d\omega$$

$$= \frac{b}{2\pi a} \int_{-a}^{a} (a - |\omega|)(\cos\omega\tau + i\sin\omega\tau) \, d\omega$$

$$= \frac{b}{2\pi a} \, 2\int_0^a (a-\omega) \cos \omega\tau \, d\omega$$

$$= \frac{b}{\pi a} \left[(a-\omega) \frac{\sin \omega\tau}{\tau} - \frac{\cos \omega\tau}{\tau^2} \right]_0^a$$

$$= \frac{b}{a\pi} \left[\frac{-1}{\tau^2} \cos a\tau + \frac{1}{\tau^2} \right]$$

$$R_{XX}(\tau) = \frac{b}{a\pi\tau^2} [1 - \cos a\tau]$$

5]. The power spectral density of zero mean WSS Process $X(t)$ is given by $S_{XX}(\omega) = \begin{cases} 1, & |\omega| < a \\ 0, & \text{otherwise.} \end{cases}$ Find the autocorrelation function & show that $X(t)$ & $X(t + \frac{\pi}{a})$ are uncorrelated.

Soln. Given $E[X(t)] = 0$

i). Autocorrelation:

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \, e^{i\omega\tau} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-a}^{a} 1 . e^{i\omega\tau} \, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{i\omega\tau}}{i\tau} \right]_{-a}^{a} = \frac{1}{2\pi} \left[\frac{e^{ia\tau} - e^{-ia\tau}}{i\tau} \right]$$

$$= \frac{1}{\pi\tau} \left[\frac{e^{ia\tau} - e^{-ia\tau}}{2i} \right]$$

$$R_{XX}(\tau) = \frac{1}{\pi i} \sin a\tau$$

ii). To prove: $X(t)$ & $X(t + \pi/a)$ are uncorrelated

ie., $\operatorname{Cov}[X(t) \, X(t + \pi/a)] = 0$

we know that $\operatorname{cov}(X, Y) = R_{XX}(\tau) - E(X)\,E(Y)$

$\because \operatorname{Cov}\left(X(t)\, X\left(t+\frac{\pi}{a}\right)\right) = R_{XX}(\pi/a) - E[X(t)]\,E[Y(t)]$

$$= R_{XX}(\pi/a) - 0$$

$$= \frac{1}{\pi \cdot \pi/a} \sin a(\pi/a)$$

$$= \frac{a \sin \pi}{\pi^2}$$

$$= 0$$

$\therefore$ $X(t)$ & $X(t+\pi/a)$ are uncorrelated