

Correlation and Spectral Densities

Auto Correlation:

Auto correlation of a random process $\{x(t)\}$ is defined by $R_{xx}(\tau) = E[x(t)x(t+\tau)]$

Properties:

i. The mean square value of the random process may be obtained from the auto correlation function $R_{xx}(\tau)$, by putting $\tau=0$.

$$\text{i.e., } E[x^2(t)] = R_{xx}(0)$$

Proof:

$$\text{WKT, } R_{xx}(\tau) = E[x(t)x(t+\tau)]$$

Put $\tau=0$,

$$R_{xx}(0) = E[x(t)x(t)]$$

$$R_{xx}(0) = E[x^2(t)]$$

ii. $R_{xx}(\tau)$ is an even function of τ .

$$\text{i.e., } R_{xx}(\tau) = R_{xx}(-\tau)$$

Proof:

$$\text{WKT } R_{xx}(\tau) = E[x(t)x(t+\tau)]$$

Replace τ by $-\tau$

$$R_{xx}(-\tau) = E[x(t)x(t-\tau)]$$

$$\text{Now } t-\tau = t_1 \Rightarrow t = \tau + t_1$$

$$\therefore R_{xx}(-\tau) = E[x(\tau + t_1)x(t_1)]$$

$$= E[x(t_1)x(t_1 + \tau)]$$

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

3]. The maximum value of $R_{xx}(\tau)$ is attained at the point $\tau=0$

$$\text{i.e., } |R_{xx}(\tau)| \leq R_{xx}(0)$$

4]. If the process $x(t)$ contains a periodic component then $R_{xx}(\tau)$ will also contain periodic component of the same period.

5]. $\lim_{|\tau| \rightarrow \infty} R_{xx}(\tau) = \bar{x}^2$

Problems :

1]. Check whether the following functions are valid autocorrelation functions.

i). $R_{xx}(\tau) = \cos \tau + \frac{|\tau|}{T}$ [Valid]

ii). $\exp(\pi\tau)$ [Not valid]

iii). $\tau^2 + \tau^2$ [Not valid]

iv). $\frac{1}{1+9\tau^2}$ [Valid]

Soln.
i). $R_{xx}(-\tau) = \cos(-\tau) + \frac{|-\tau|}{T} = \cos \tau + \frac{|\tau|}{T} = R_{xx}(\tau)$

2]. Determine the mean & variance of the process given that the autocorrelation function

$$R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

Soln.

By property, $R_{xx}(0) = E[x^2(t)]$

Now, $R_{xx}(0) = 25 + \frac{4}{1+0} = 29 = E[x^2(t)]$

$$\text{and } \lim_{|\tau| \rightarrow \infty} R_{xx}(\tau) = \lim_{|\tau| \rightarrow \infty} \left[25 + \frac{4}{\tau^2} \right]$$

$$= 25 = E[x(t)]^2$$

$$\Rightarrow E[x(t)] = 5, \quad E[x^2(t)] = 29$$

$$\text{Var}[x(t)] = E[x^2(t)] - \{E[x(t)]\}^2 = 29 - 25$$

$$\text{Var}[x(t)] = 4$$

3]. A Stationary random process has an auto correlation function and is given by,

$$R_{xx}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4} \text{ . find the mean \& variance}$$

Soln. By Properties,

$$i). \bar{x}^2 = \lim_{|\tau| \rightarrow \infty} R_{xx}(\tau)$$

$$= \lim_{|\tau| \rightarrow \infty} \frac{25\tau^2 + 36}{6.25\tau^2 + 4} = \lim_{|\tau| \rightarrow \infty} \frac{\tau^2 \left[25 + \frac{36}{\tau^2} \right]}{\tau^2 \left[6.25 + \frac{4}{\tau^2} \right]}$$

$$= \lim_{|\tau| \rightarrow \infty} \frac{25 + \frac{36}{\tau^2}}{6.25 + \frac{4}{\tau^2}}$$

$$= \frac{25}{6.25}$$

$$\bar{x}^2 = 4$$

$$\bar{x} = 2 = E[x(t)]$$

$$ii). E[x^2(t)] = R_{xx}(0)$$

$$= \frac{25(0) + 36}{6.25(0) + 4} = 9$$

$$\therefore \text{Var}[x(t)] = E[x^2(t)] - [E[x(t)]]^2 = 9 - 4 = 5$$