3). Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = \log x \sin(\log x)$

Soln.

Given $[x^2D^2 + xD + 4]\, y = \log x \sin(\log x)$ ↳ (1)

Take

$x = e^z$

$\log x = z$

$xD = D'$; $x^2D^2 = D'(D'-1)$

$= D'^2 - D'$

(1) $\Rightarrow$ $[D'^2 - D' + D' + 4]y = z \sin z$

$[D'^2 + 4]y = z \sin z$

AE

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

$CF = A \cos 2z + B \sin 2z$

$PI = \frac{1}{D'^2+4} z \sin z$

$= z \frac{1}{D'^2+4} \sin z - \frac{2D'}{(D'^2+4)^2} \sin z$

$= z \frac{1}{-1+4} \sin z - \frac{2 \cos z}{(-1+4)^2}$ $\quad D'^2 \rightarrow -a^2 = -1^2 = -1$

$= \frac{z \sin z}{3} - \frac{2 \cos z}{9}$

The Soln. is

$y = CF + PI$

$y = A \cos 2z + B \sin 2z + \frac{z \sin z}{3} - \frac{2 \cos z}{9}$

$= A \cos 2(\log x) + B \sin 2(\log x)$

$+ \frac{\log x \cdot \sin(\log x)}{3} - \frac{2}{9} \cos(\log x)$

HJ. Solve $(x^2 D^2 - xD + 1)y = \log x$

Soln.

Given $(x^2 D^2 - xD + 1)y = \log x. \rightarrow (1)$

Take $x = e^z$

$z = \log x$

$xD = D'$

$x^2 D^2 = D'(D' - 1) = D'^2 - D'$

$(1) \Rightarrow \quad (D'^2 - D' - D' + 1)\, y = z$

$(D'^2 - 2D' + 1)\, y = z$

AE $\quad m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

$\therefore CF = (A + Bz)\, e^z$

$$PI = \frac{1}{D'^2 - 2D' + 1}\, z$$

$= [1 + (D'^2 - 2D')]^{-1}\, z$

$= [1 - (D'^2 - 2D') + (D'^2 - 2D')^2 - \cdots]\, z$

$= z - D'^2 z + 2D'(z)$

$PI = z + 2$

$\therefore$ The Soln. is, $\quad y = CF + PI$

$y = (A + Bz)\, e^z + z + 2$

$y = (A + B\log x)\, x + \log x + 2$