Method of Variation of Parameters

The second order linear differential eqn. is

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q = X$ where X is a fn. of x.

CF $= C_1 f_1 + C_2 f_2$, where C_1, C_2 are constants, f_1, f_2 are functions of x.

PI $= Pf_1 + Qf_2$

where $P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx$

$Q = \int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx$

$\int \tan x\,dx = +\log(\sec x)$

$\int \cot x\,dx = \log(\sin x)$

$\int \operatorname{cosec} x\,dx = -\log[\operatorname{cosec} x + \cot x]$

$\int \sec x\,dx = \log(\sec x + \tan x)$

1. Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ using method of variation of parameters.

Soln.

Given $(D^2+4)y = 4\tan 2x$ where $X = 4\tan 2x$

AE $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

CF $= C_1 \cos 2x + C_2 \sin 2x$

PI $= Pf_1 + Qf_2$

Here $f_1 = \cos 2x$, $f_2 = \sin 2x$

$f_1' = -2\sin 2x$, $f_2' = 2\cos 2x$

Now $w = f_1 f_2' - f_2 f_1'$

$= \cos 2x\,[2\cos 2x] - \sin 2x\,(-2\sin 2x)$

$= 2\cos^2 2x + 2\sin^2 2x$

$= 2[\cos^2 2x + \sin^2 2x]$

$= 2(1) = 2$

$$P = -\int \frac{f_2 x}{w}\, dx$$

$$= -\int \frac{\sin 2x \cdot 4\tan 2x}{2}\, dx$$

$$= -2\int \sin 2x \cdot \frac{\sin 2x}{\cos 2x}\, dx$$

$$= -2\int \frac{\sin^2 2x}{\cos 2x}\, dx$$

$$= -2\int \frac{1-\cos^2 2x}{\cos 2x}\, dx = -2\left[\int \frac{1}{\cos 2x}\, dx - \int \cos 2x\, dx\right]$$

$$= -2\left[\int \sec 2x\, dx - \int \cos 2x\, dx\right]$$

$$= -2\left[\frac{\log(\sec 2x + \tan 2x)}{2} + \frac{\sin 2x}{2}\right]$$

$$P = -\log(\sec 2x + \tan 2x) + \sin 2x$$

$$Q = \int \frac{f_1 x}{w}\, dx$$

$$= \int \frac{\cos 2x \cdot 4\tan 2x}{2}\, dx$$

$$= 2\int \cos 2x \cdot \frac{\sin 2x}{\cos 2x}\, dx$$

$$= 2\int \sin 2x\, dx = 2\left[-\frac{\cos 2x}{2}\right]$$

$$Q = -\cos 2x$$

$$PI = Pf_1 + Qf_2$$

$$= [-\log(\sec 2x + \tan 2x) + \sin 2x]\cos 2x - \cos 2x \sin 2x$$

$$PI = -\log(\sec 2x + \tan 2x)\cos 2x$$

$$\therefore y = CF + PI = C_1 \cos 2x + C_2 \sin 2x - \log(\sec 2x + \tan 2x)\cos 2x$$